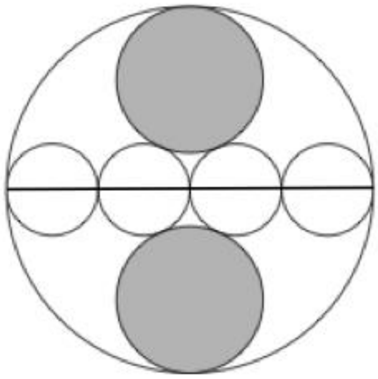
Mediterranean Youth Mathematical Championship (MYMC)

Rome, July 21, 2016

Morning round

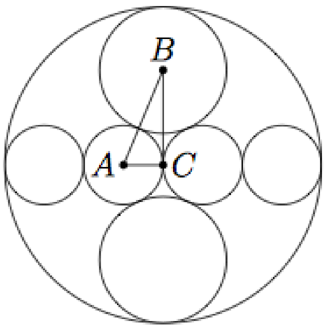
WE1.

In the figure below, all the circles are tangent; the four white small circles are equal to one another, and so are the two grey circles. The radius of the large circle is 30. Find the radius of the grey circles.



Solution

The answer is 12. Let r be the radius we are looking for. Let A, B, C be the centers of the circles, as in the figure below.



Using the Pythagorean Theorem, $AC^2 + BC^2 = AB^2$, where $AC = 15/2$, $BC = 30 - r$, $AB = r + 15/2$, we get

$$(15/2)^2 + (30 - r)^2 = (r + 15/2)^2.$$

Expanding and simplifying we can solve for r and obtain $r = 12$.

WE2. (Leonardo Pisano, *Liber Abbaci*, 1202)

A man buys 30 birds for 30 denari. He is able to choose from partridges, pigeons, and sparrows [which are three types of birds; he chooses at least one bird for each type]. A partridge costs 3 denari, a pigeon costs 2 denari, and 2 sparrows cost 1 denaro (i.e. 1 sparrow costs ½ a denaro). Work out how many of each bird the man buys.

<u>Solution</u>

The answer is (3, 5, 22). Let x, y and z be the number of partridges, pigeons and sparrows, respectively. We have $3x + 2y + (1/2)z = 30$ and $x + y + z = 30$. Since $z = 30 - (x + y)$, we get: $3x + 2y + 15 - (1/2)x - (1/2)y = 30$. Collecting like terms, we get $(5/2)x + (3/2)y = 15$, i.e., $5x + 3y = 30$. Both $3y$ and 30 are multiples of 3, so also $5x$ must be a multiple of 3. The only possibility is $x = 3$, since y cannot be 0.

Hence the answer is: 3 partridges, 5 pigeons, and $30 - 3 - 5 = 22$ sparrows.

WE3.

The inhabitants of a distant island are of two kinds: knaves, who always lie, and knights, who always tell the truth. One day, an explorer meets 6 inhabitants, who say the following sentences, in this order:

X: «we are all of the same kind, except one»

Y: «what *X* said is false»

U: «we are all of the same kind, except two»

V: «what *U* said is false»

W: «there are three of us of one kind and three of the other»

Z: «what *W* said is false».

The explorer can deduce that:

A) there is exactly one knave, but it is not possible to determine who it is

B) there is exactly one knave and it is possible to determine who it is

C) there are exactly two knaves, but it is not possible to determine who they are

D) there are exactly two knaves and it is possible to determine who they are

E) there are exactly three knaves, but it is not possible to determine who they are

F) there are exactly three knaves and it is possible to determine who they are

<u>Solution</u>

The answer is F). The first observation is that *X* and *Y* are of different kinds; the same is true for *U* and *V*, and also for *W* and *Z*. Therefore, among the 6 inhabitants, there must be 3 knights and 3 knaves. At this point it is easy to conclude that the knaves are *X*, *U*, *Z*.

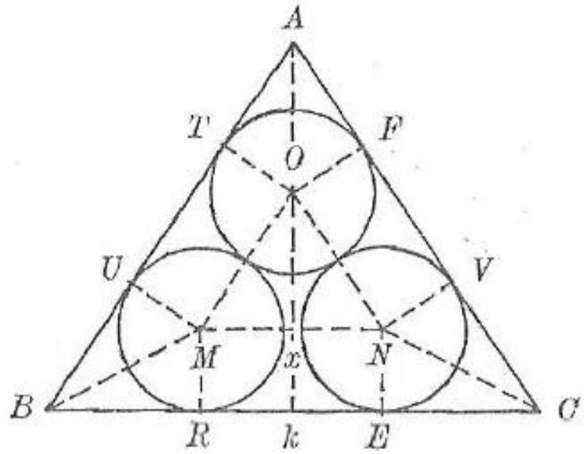
WE4. (Piero della Francesca, *Trattato d'abaco*, around 1470)

In a triangle *ABC* the sides are such that *AB* = *AC* = 80, *BC* = 96. Three equal circles are placed inside the triangle; they are as large as possible, but they do not overlap (the circles can be tangent to each other and to the sides of the triangle). What is the diameter of the circles?

Solution

[We add the original text and figure:

Egl'è uno triangulo ABC, del quale il lato AB è 80 et il lato AC è pure 80 et il lato BC è 96; nel quale triangulo voglo mettere 3 tondi e' magiori che se po'. Domando quanto sirà il loro diametro.]



The answer is 30. Firstly, we observe that each circle is tangent to two sides of the triangle, and that the two circles that are tangent to the longer side *BC* are not mutually tangent (otherwise the triangle would be equilateral). Let *x* be the radius of the circles; the diameter we are looking for must therefore be 2*x*.

The sides of the right triangle *AkB* are 48, 64, 80; the triangle *ATO* is similar to *AkB*, and therefore its legs measure *x* and 4*x*/3.

The triangle with vertices at the centers of the circles is similar to the original triangle, because the sides of the two triangles are pairwise parallel: therefore, *NO* = *OM* = 2*x* and *MN* = 12*x*/5.

We can deduce that *BR* = (96 − 12*x*/5)/2 = 48 − 6*x*/5.

The side *AB*, whose measure is 80, is made up of the sum *AT* + 2*x* + *UB*, where *UB* = *BR*. Thus

$$80 = 4x/3 + 2x + 48 - 6x/5$$

from which 32*x*/15 = 32 and *x* = 15.

WE5.

It is simple to check that $2016 = (2^{6-1}) \cdot (2^6 - 1)$. Some other years can be written in the form $(n^{m-1}) \cdot (n^m - 1)$, with *n*, *m* positive integers strictly greater than 1. What will the next year like this be after 2016?

Solution

The answer is $2160 = (3^{4-1}) \cdot (3^4 - 1)$. Note that for each value of m there is a minimum integer n such that $(n^{m-1}) \cdot (n^m - 1) > 2016$.

For instance, for $m = 2$ the minimum is $13 \cdot (13^2 - 1)$; for $m = 3$, the minimum is $5^2 \cdot (5^3 - 1)$; for $m = 5$, the minimum is $3^4 \cdot (3^5 - 1)$; for $m = 7$, the minimum is $2^6 \cdot (2^7 - 1)$.

It is easy to check that all these values are greater than $2160 = 3^3 \cdot (3^4 - 1)$, that corresponds to $m = 4$.

WE6.

In the Olympus Athena and Poseidon want to play the following game.

A part of the Mediterranean Sea can be thought of as a rectangular board with $n \times 10$ squares. The ship of Odysseus is placed in the north-east corner and has to reach the south-west corner. Athena starts and each player can ask Aeolus to blow wind to move the ship either west or south by 1 or 2 squares, and Aeolus always fulfills their request. The player who cannot move (because the previous move makes Odysseus' ship reach the south-west corner) loses.

Athena and Poseidon have yet to agree on the number n, but they have decided that $1 \leq n \leq 10$. How many values for n within this range are winning numbers for Athena, that is, are numbers such that the first player has a winning strategy?

Solution

The answer is 6. Let us identify squares by the numbers of their row and column, so that the south-west square corresponds to (1, 1) and the north-east square to (n, 10).

Consider the set L of all squares (a, b) such that $a - b$ is a multiple of 3 (note that the south-west corner belongs to L). Now it is easy to see that:

1) if Odysseus' ship is in one of the squares in L and a player X moves it, it will end up on a square outside of L, since there are no two squares of L reachable one from the other in a single move;

2) if Odysseus' ship is on one of the squares outside of L, it can be moved to a square of L in a single move.

In particular, if a player moved the ship starting from a square in L across h squares in one direction, the next player can move it back to L by moving it across h squares in the other direction, or across $3 - h$ squares in the same direction; and at least one of these possibilities will always be available. Therefore,

a) if the north-east square is not in L the first player can move the ship to a square in L and then always make the second player move from a square in L. Since the sum of the row number and the column number decreases at every move, the ship will eventually end up in the south-west corner with the last move of the first player, who will therefore win the game;

b) otherwise, the same strategy can be used by the second player to win.

There are 4 numbers n between 1 and 10 such that the first square is in L: 1, 4, 7, 10. The first player wins in all the 6 other cases.

WE7.

An equilateral triangle T_1 is circumscribed to a circle, while an equilateral triangle T_2 is inscribed in the same circle. What is the ratio between the areas of T_1 and T_2?

A) 2

B) 3

C) 4

D) 9

E) 12

Solution

The answer is C). The ratio is 4, because a side of T_2 is half of a side of T_1.

WE8.

ABCD is a quadrilateral with parallel sides *AB* and *CD*; let these sides be such that $AB > CD$. Moreover $DA = AB = BC$. In this case the diagonals *AC* and *DB*:

A) are mutually perpendicular

B) are bisectors of the angles with vertices in *A* and *B*

C) are bisectors of the angles with vertices in *C* and *D*

D) mutually meet at their midpoints

E) each has length equal to half the perimeter of the quadrilateral

Solution

The answer is C). Two isosceles triangles can be identified: *ABD* and *BAC*; each has one of the quadrilateral's diagonals as a base. So the angles $\angle CAB$ and $\angle ACB$ are equal; also the angles $\angle DCA$ and $\angle CAB$ (alternate interior angles of parallel lines) are equal. Therefore $\angle DCA = \angle ACB$ and *AC* is the bisector of the angle with vertex *C*. A similar argument holds for the diagonal *BD*. It is easy to check that the other statements in general are false.

WE9.

How many positive numbers with 4 digits are multiples of 3, have 2 as their first digit (the thousands digit), and 8 as their last digit (the units digit)?

Solution

The answer is 33. Numbers such as the ones required can be written as $2ab8$, where a and b are digits between 0 and 9. Since the number is a multiple of 3, the sum $2 + a + b + 8$ is a multiple of 3; in other words, $1 + a + b$ is a multiple of 3.

If a is 2, or 5, or 8, there are 4 possible values for b (that is, 0, 3, 6, 9); for any other digit a, there are 3 possible values of b. The answer is $3\cdot4 + 7\cdot3$.

WE10.

There are three people X, Y, Z; each of them secretly communicates to a judge a positive integer less than or equal to 10. The judge, in front of all three people, says that the sum of the three numbers is 26; so the judge asks X, Y, Z what the product of the three numbers is.

X says he cannot determine the product. Also Y, who heard X's answer, says he cannot determine the product. Finally Z, who heard the preceding answers, says that neither he can determine the product.

At this point, after having all heard the others' answers, the judge asks X, Y, Z to write down the product of the three numbers; each person has to write down the product without communicating with the others.

Who will be able to answer correctly?

A) Only X

B) Only Y

C) Only Z

D) Only X and Y

E) Only Y and Z

F) Only X and Z

Solution

The answer is D). Firstly we notice that if the sum of the three numbers is 26, then each one is greater than or equal to 6. However, if one of the people has communicated to the judge the number 6, then that person could easily know the product (600); the same is true if one person has initially communicated to the judge the number 7, because the product can be uniquely determined ($7\cdot9\cdot10 = 630$). Therefore, each person knows that the others communicated the numbers 8, or 9, or 10.

Let us call x, y, z the numbers initially communicated by X, Y, Z.
If $z = 10$, then, since $x + y = 16$, Z would know that $x = y = 8$; analogously, if $z = 9$, then, since $x + y = 17$, Z would know that the product is $9 \cdot 8 \cdot 9$.
Following this line of reasoning, X and Y are able to conclude that $z = 8$. Both X and Y, also knowing their own number, can therefore correctly write the product. On the other hand, Z, after having answered that he was not able to determine the product the first time, has no additional information.

WE11.

An urn contains white balls and black balls, all of the same size; the total number of balls in the urn is less than 100. If 2 balls are drawn (that is, removed from the urn without putting the first one back into the urn before drawing the second ball), the probability of drawing 2 white balls is 1/2016. How many white balls and black balls are in the urn before any are drawn? [You are required to give a solution; it can be proven that there is only one solution.]

<u>Solution</u>

The answer is $(w; b) = (2; 62)$, where w is the number of white balls and b the number of black balls (this is the only solution such that $w + b < 100$). The number of possible cases is $\binom{w+b}{2}$, while the number of favorable cases is $\binom{w}{2}$. Therefore we have

$$(w + b)(w + b - 1) = 2016\, w(w - 1) \qquad (*)$$

We can search, through successive trials, whole solutions starting from small values of w, and keeping in mind that $2016 = 2^5 \cdot 3^2 \cdot 7 = 32 \cdot 63$ and that the left hand term of (*) is the product of two consecutive numbers. For $w = 2$ the right hand term of (*) is $2 \cdot 32 \cdot 63 = 64 \cdot 63$; therefore one solution can be immediately determined: $w = 2$ and $b = 62$.

WE12.

Find the unique integer number n, with $1 < n < 1000$, such that n^2 equals the cube of the sum of the digits of n.

<u>Solution</u>

The answer is 27. In order for n^2 to be a cube, n must be a cube itself. Since $1 < n < 1000$, n equals 2^3 or 3^3 ... or 9^3. Now, the sum of the digits of n is at most $9 + 9 + 9 = 27$. For $n \geq 6^3$, we have $n^2 \geq 6^6 > 27^3$, which is too large. So the only possibilities are: 8, 27, 64, 125; and it can be easily verified that 27 has the required property; this is not true for the other cubes.

WE13.

During a certain range of days the weather conditions were observed, and it turned out that if in the morning it was raining, in the afternoon it was clear. In the same range of days there were 18 days in which it rained, 14 clear mornings and 12 clear afternoons. How many days were there in the range of days observed?

A) 18

B) 22

C) 24

D) 26

E) 28

Solution

The answer is B). If after a rainy morning there is always a clear afternoon, it is not possible that on a same day it rains both in the morning and in the afternoon. Therefore we can simply count half days. There are 14+12 clear half days; of these, 18 are half days of days in which it rained (during the other half of the day). So there are 14+12–18 = 8 half days left, that is 4 days in which it never rained. Therefore the range of days during which the weather was observed is of 4+18 = 22 days.

WE14.

The three vertices of a right triangle are placed on two sides and at a vertex of a rectangle as shown in the figure below. Knowing the lengths marked in the figure, find the area of the rectangle.

20

12

7

(The figure is not drawn proportionally with respect to the lengths of the marked segments.)

Solution

The answer is 384. The length of the base of the rectangle can easily be found: it is 16; so the right triangle in the lower left hand corner has a leg of length 9. Now we can proceed in different ways; one possibility is to notice that the right triangles on the left of the figure are similar (the sum of the measures of the angles adjacent to the marked right angle is 90°). From this we can deduce that the height of the rectangle has length 12 + 12 = 24; so the area of the rectangle is 24·16 = 384.

WE15.

The polynomial $p(x)$ is of degree 3 and its coefficients are all integers. Which of the following numbers is necessarily a divisor of $p(100) - p(1)$?

A) 25

B) 100

C) 16

D) 49

E) 33

Solution

The answer is E). Let $q_n(x)$ be $a_n x^n$ (where a_n is an integer). For any n the difference $(100 - 1)$ is a divisor of $q_n(100) - q_n(1) = a_n 100^n - a_n$. As a consequence, $99 = 100 - 1$ is a divisor of $p(100) - p(1)$.

Note that the degree of the polynomial $p(x)$ is of no interest.