Mediterranean Youth Mathematical Championship (MYMC)

Rome, July 21, 2016

Afternoon round – First stage

RE1A.

What are the last two digits of 2016^{2016}?

A) 16

B) 36

C) 56

D) 76

E) 96

Solution

The answer is A). It is enough to consider the last two digits of the base 2016 (the initial two digits "20" have no effect on the last two digits of the result). Similarly, when considering the successive powers of 16, every time we can limit our attention to the last two digits (technically, we would say that we can operate *modulo* 100). We have:

$16^1 = 16$, $16^2 = ...56$, $16^3 = ...96$, $16^4 = ...36$, $16^5 = ...76$, $16^6 = ...16$

Therefore, we have also

$16^{11} = ...16$, . . . , $16^{16} = ...16$, . . . , $16^{2016} = ...16$.

RE1B.

An urn contains an even number of balls; half of the balls in the urn are white and the other half black. If three balls are drawn at the same time from the urn, the probability of drawing 3 balls of the same color is 20%. How many balls are there in the urn before any balls are drawn?

Solution

The answer is 16. Let us suppose that the urn contains n white balls and n black balls. Let us look at the drawn balls one at a time. Whatever the color of the first is, for the event "the three extracted balls are of the same color" there are $(n-1)(n-2)$ favorable cases, while the possible cases are $(2n-1)(2n-2) = 2(2n-1)(n-1)$. Therefore we get:

$$\frac{(n-1)(n-2)}{2(2n-1)(n-1)} = \frac{1}{5}$$

that is $5(n-2) = 2(2n-1)$.

We get $n = 8$; so in the urn there are initially 16 balls before any are drawn.

RE1C.

A pyramid *VABC* has triangle *ABC* as its base; the three side edges *VA*, *VB*, *VC* have lengths 12, 12, 14 and they are pairwise perpendicular. Determine the length of the radius of the spherical surface passing through points *V*, *A*, *B*, *C*.

Solution

The answer is 11. Let us consider the parallelepiped determined by the edges *VA*, *VB*, *VC*: the sphere circumscribed to the pyramid is also circumscribed to the parallelepiped. One diagonal of the parallelepiped is equal to the diameter of the sphere, whose length is therefore $\sqrt{12^2 + 12^2 + 14^2} = 22$.

One can also think of a Cartesian system having origin at *V* and axes *VA*, *VB*, *VC*. The center of the sphere passing through *V* and *A*, *B*, *C* lies on the planes that are respectively perpendicular to the segments *VA*, *VB*, *VC* and passing through their midpoints. The equations of the planes are $x = 6$, $y = 6$, $z = 7$; the coordinates of the center are (6, 6, 7), and the radius is easily found.

GE1A. (Leonardo Pisano, *Liber Abbaci*, 1202)

There are two ships some distance apart. The first ship can cover the distance between them in 5 days, while the second ship can cover the same distance in 7 days. If the two ships begin the journey towards each other at the same time, in how many days will they meet?

A) more than 4 days

B) 4 days

C) between 3 and 4 days

D) 3 days

E) less than 3 days

Solution

The answer is E). Let *d* be the distance and let *x* be the number of days required. The speeds of the ships are $d/7$ and $d/5$. Then $(d/7) + (d/5) = d/x$. Dividing by *d* and multiplying by *x*, we find that $x/7 + x/5 = 1$. Therefore $(5+7)x = 35$, so $x = 35/12$.

GE1B.

If a pyramid and a prism have the same number of edges, then the pyramid has:

A) more faces and more vertices than the prism

B) fewer faces and fewer vertices than the prism

C) the same number of vertices and of faces as the prism

D) more faces and fewer vertices than the prism

E) fewer faces and more vertices than the prism

Solution

The answer is D). A pyramid with an n-gonal base has $2n$ edges, $n+1$ faces and $n+1$ vertices; a prism with an n-gonal base has $3n$ edges, $n+2$ faces and $2n$ vertices. Since the number of the edges in a pyramid is even and in a prism it is a multiple of 3, we can assume that the number is $6k$, with $k \geq 2$. So the pyramid has $3k+1$ faces and $3k+1$ vertices, while the prism has $2k+2$ faces and $4k$ vertices. At this point it is enough to observe that if $k \geq 2$, then $3k+1$ is greater than $2k+2$ and less than $4k$.

GE1C.

Find the smallest positive integer $n \geq 2$ such that none of the fractions

$$\frac{19}{n+19} \quad \frac{20}{n+20} \quad \frac{21}{n+21} \quad \dots \quad \frac{91}{n+91}$$

can be simplified.

Solution

The answer is 97. The difference between the denominator and the numerator of each fraction is n. Since the GCD $(19, n + 19)$ = GCD $(19, n)$ and analogous considerations hold for the other fractions, n must be co-prime with all integers from 19 to 91. We conclude that n must have only prime factors larger than 91. The smallest integer satisfying this condition is 97.