Mediterranean Youth Mathematical Championship (MYMC)

Rome, July 21, 2016

Afternoon round – Last stage

RE2A.

A regular hexagon and an equilateral triangle have the same perimeter. The ratio between the area of the triangle and the area of the hexagon is:

A) 1:1

B) 1:2

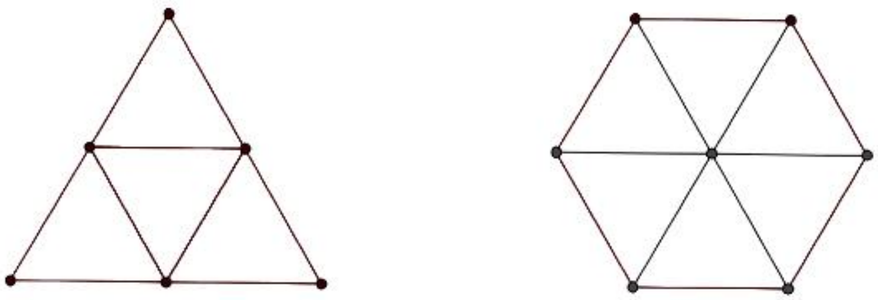
C) 2:3

D) 3:5

E) 5:6

Solution

The answer is C). If the perimeters are the same, each side of the triangle must be double the side of the hexagon. The triangle can be partitioned into 4 equilateral triangles, each equal to each of the 6 equilateral triangles that make up the hexagon. Therefore the ratio between the areas is 4:6 = 2:3.



RE2B.

An infinite sequence of positive integers $x_1, x_2, \dots, x_n, \dots$ satisfies the relation: $(x_n)^2 = (4n + 9) + (n - 4)\, x_{n+1}$. Find x_1.

Solution

The answer is 2. Since $n - 4 = 0$ for $n = 4$, we can consider $n = 4$ and rewrite the relation as $x_4^2 = (4 \cdot 4 + 9) + 0 = 25$, that is $x_4 = 5$. Knowing x_4, the relation now allows us to calculate x_3 : indeed, $x_3^2 = (4 \cdot 3 + 9) + (3 - 4)\, x_4 = 21 - 5 = 16$, that is $x_3 = 4$. Continuing in the same way we find $x_2 = 3$ and finally $x_1 = 2$. Notice that, in general, it could be $x_n = n + 1$, since the following identity is true:

$$(n + 1)^2 = (4n + 9) + (n - 4)(n + 2).$$

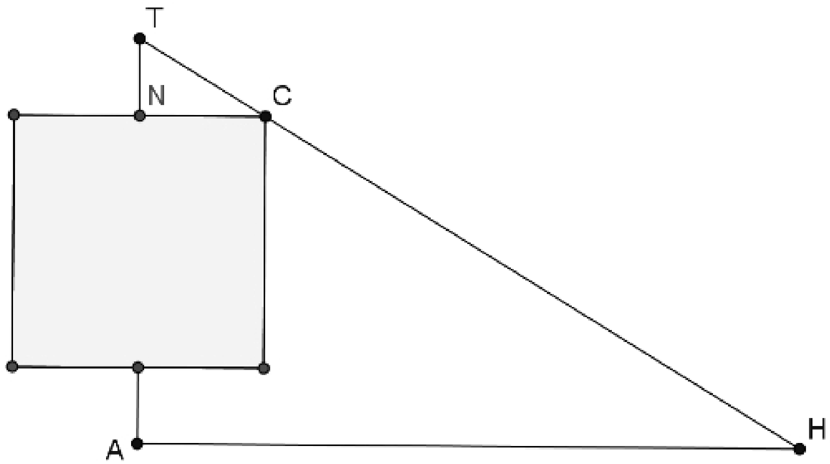
RE2C. (*Chiu Chang Suan Shu*, circa 250 BC)

A city is surrounded by square shaped walls oriented towards north, east, south, and west. There are two doors, one in the middle of the north side and one in the middle of the south side. Leaving the city from the north door and walking straight towards north, after 80 steps there stands a tree. If you leave from the south door, walking straight towards south, and then after 80 steps turn 90°, the same tree appears after walking straight another 600 steps. What is the length [in steps] of the side of the city?

Solution

The answer is 240. Refer to the figure, where N is the north door and T is the tree. The triangles TNC and TAH are similar. Call x the length of the side of the city, you have:

$TN : NC = TA : AH$ that is $80 : (x/2) = (80 + x + 80) : 600$



You get, for the positive solution, $x = -80 + 320 = 240$.

GE2A.

Two circles with the same radius intersect at points A and B. Each circle passes through the center of the other one, and $AB = 10\sqrt{3}$ m. What is the area of the "8" shaped figure made up of the two circles together?

A) $\frac{400}{3}\pi + 50\sqrt{3}$ m²

B) $\frac{500}{3}\pi + 50\sqrt{3}$ m²

C) 200π m²

D) $\frac{500}{3}\pi$ m²

E) $50\sqrt{3}$ m²

Solution

The answer is A). Let O and O' be the centers of the circles. In the rhombus made up of the 2 equilateral triangles AOO' and BOO', the side is the radius of the circles and the longer diagonal is AB. The area we are looking for is 2/3 of the area of each circle plus the area of the rhombus. The radius of the circles is 10 m, and so the area of the rhombus is $\frac{10 \cdot 10\sqrt{3}}{2} = 50\sqrt{3}$ m^2.

The area of the "8" shaped figure is

$$2 \cdot \frac{2}{3} \cdot 100\pi + 50\sqrt{3} = \frac{400}{3}\pi + 50\sqrt{3} \text{ (in m}^2\text{)}.$$

GE2B. (Leonardo Pisano, *Liber Abbaci*, 1202)

A certain man entered a certain pleasure garden through 7 doors, and he took from there a number of apples; when he wished to leave he had to give the first doorkeeper half of all the apples and one more; to the second doorkeeper he gave half of the remaining apples and one more. He gave to the other 5 doorkeepers similarly, and there was one apple left for him. It is sought how many apples there were that he collected.

Solution

The answer is 382. Leaving the garden the man has one apple; so before he passed the 7$^{\text{th}}$ door he had $4 = 2 \cdot (1 + 1)$ apples. Before he passed the 6$^{\text{th}}$ door he had $2 \cdot (4 + 1)$ apples. In general, if after passing a door the man has R apples, before passing that door he had $2 \cdot (R + 1)$ apples.

Therefore we can reconstruct the number of apples the man had initially: 1, 4, 10, 22, 46, 94, 190, 382.

GE2C.

Consider a deck of 63 cards, different from one another. Define a *shuffle* on the deck as follows: take the first 31 cards of the deck and place them one after the other in between each pair of the remaining 32 cards. After a shuffle, you obtain a new arrangement of the deck.

How many times does the shuffle have to be repeated in order to return to the initial arrangement of the cards in the deck?

Solution

The answer is 6. After the first shuffle the "distance" between two cards which were consecutive in the original deck is 2, because a new card has been placed between them. After the second shuffle the distance between two consecutive cards in the original deck is 4, because two other cards have been placed between them.

In general, after each shuffle, the distance between two consecutive cards in the original deck doubles; so we obtain the powers of 2. To solve the problem, we need to find the least positive power n of 2 such that $2^n - 1$ is a multiple of 63. Of course $2^6 = 64 = 1 + 63$.