Mediterranean Youth Mathematical Championship (MYMC)

Rome, July 21, 2016

Afternoon round – Intermediate stage

RE3A. (Leonardo Pisano, *Liber Abbaci*, 1202)

A man sells two pieces of gold, which together weigh one pound [= 12 ounces]. He sells the first piece of gold at the price of 68 bezants per pound, and he sells the second piece at 50 bezants per pound [a bezant is a valuable coin]. In total he receives 56 bezants. What is the weight [in *ounces*] of the first piece of gold?

Solution

The answer is 4. Let x be the weight of the first piece, expressed as the fraction of 1 pound. Then $1–x$ corresponds to the weight of the second piece. Hence $68x + 50(1–x) = 56$. It follows that $18x = 6$, therefore $x = 1/3$. One third of a pound is 4 ounces.

RE3B.

Consider three equal segments $AB = BC = CD$. Suppose that points A, C, D are collinear (and, more precisely, that C is between A and D), and that the measure of angle $\angle BCD$ is three times the measure of angle $\angle ABC$. You can deduce that triangle ABD is

A) equilateral

B) isosceles but not equilateral

C) right and not isosceles

D) obtuse and not isosceles

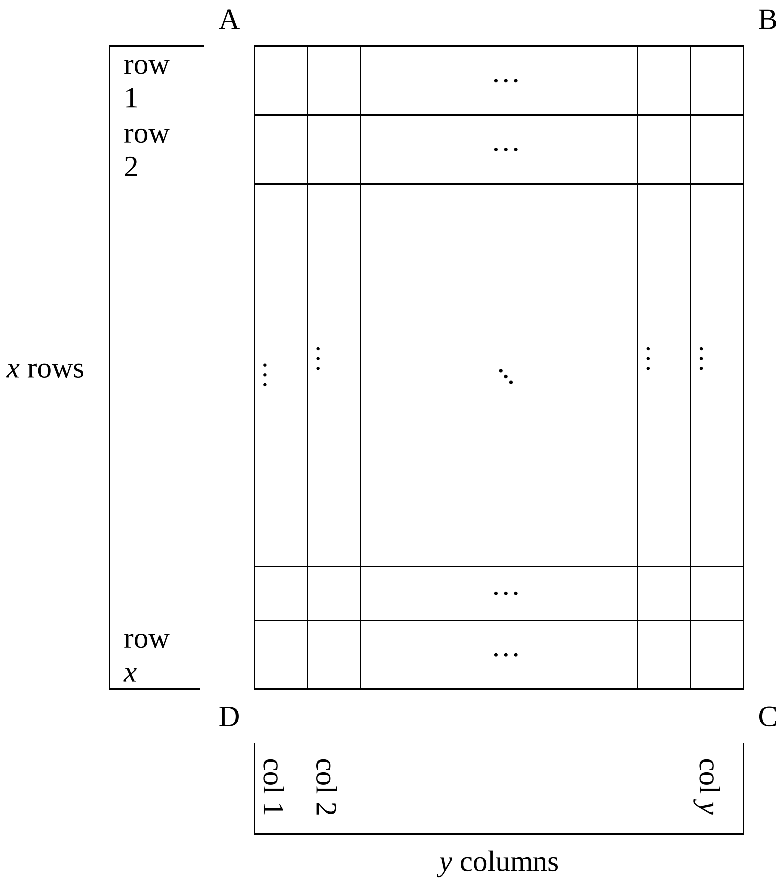
E) degenerate (that is points A, B, D are collinear).

Solution

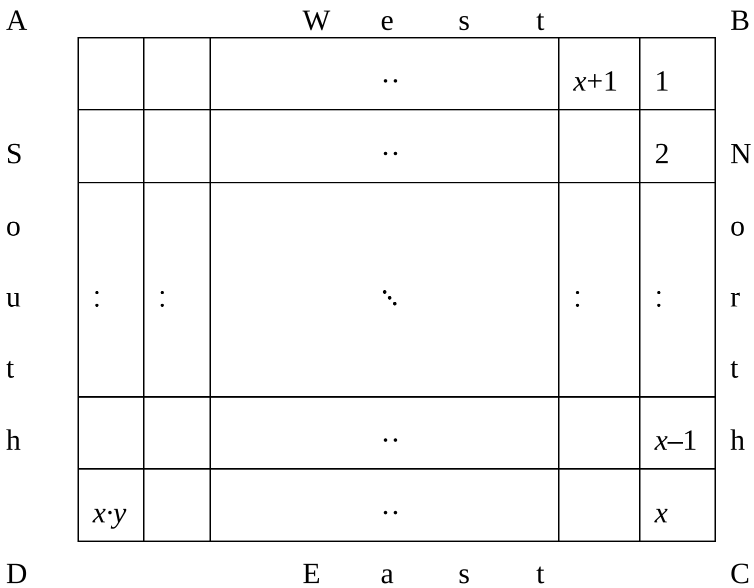
The answer is B). In triangle ABC the angles with vertices at A and at C are equal; moreover, the sum of the measure of the angle with vertex at C and three times the measure of the angle with vertex at B is 180°. It easily follows that the measure of angle $\angle ABC$ is 36°. This way we find that the measures of the angles of the triangle ABD are 72°, 72°, 36°.

RE3C.

The rectangular pavement in the figure below contains exactly $x{\cdot}y$ square tiles, with $9 \leq y \leq x$ (x is the number of rows, y is the number of columns).



The pavement has its sides oriented north, south, east and west (but we do not know which side is oriented north) and the tiles are numbered from 1 to $x{\cdot}y$, proceeding from west to east and from north to south. For example, if *BC* were the side oriented north you would have:



Knowing that the number 33 is assigned to the tile positioned on the row $x - 5$ and on the column $y - 7$, determine the total number of tiles.

<u>Solution</u>

The answer is 243.

The orientation

	N	
W		E
	S	

is not possible, for it yields $y(x-6) + (y-8) + 1 = 33$, from which $y(x-5) = 40$, but $y = 10$ and $x = 9$ violate the condition $y \leq x$. The same type of violation occurs for $y = 20$ and $y = 40$.

The orientation

	S	
E		W
	N	

is not possible, for it yields $5y+7+1 = 33$, from which $5y+8 = 33$, but $y = 5$ violates the condition $y \geq 9$.

The orientation

	W	
S		N
	E	

is not possible for it yields $7x+(x-6)+1 = 33$, from which $8x-5 = 33$, but $x = 38/8$ violates the condition that x is an integer.

The only other possible orientation is

	E	
N		S
	W	

which yields $x(y-8)+5+1 = 33$, from which $x(y-8) = 27$; since $x \geq y \geq 9$, we can only accept $x = 27$ and $y = 9$. Therefore we can conclude that the total number of tiles must be $x \cdot y = 243$.

GE3A. (Leonardo Pisano, *Liber Abbaci*, 1202)

There is a least positive integer which, when divided by 2, has remainder 1; when divided by 3, has remainder 2; when divided by 4, has remainder 3; when divided by 5, has remainder 4; when divided by 6, has remainder 5; and is exactly divisible by 7. What is the number?

<u>Solution</u>

The answer is 119. Let x be the number. The assumptions say that $x - 1$ is divisible by 2, $x - 2$ is divisible by 3, $x - 3$ is divisible by 4, $x - 4$ is divisible by 5, and $x - 5$ is divisible by 6. Equivalently: $x + 1$ is divisible by

2, 3, 4, 5, 6. Since lcm(2, 3, 4, 5, 6) = 60, the number x is the smallest positive integer of the form $-1 + 60k$ (with k a positive integer) which is divisible by 7. Since $60 = 4 + 7 \cdot 8$, this means that the solution corresponds to the smallest positive integer k such that $-1 + 4k$ is divisible by 7. Hence $k = 2$ and $x = 119$.

GE3B.

"*The magic wand*" is a poem consisting of exactly 1000 verses. The author has numbered these, in the correct order, but she dislikes the digit 1; so, in numbering, she avoids any number with 1 as a digit. Therefore the first verse is numbered 2; the second verse 3; the verse after the one numbered 9 is 20, and so on. What is the number assigned to the last verse?

<u>Solution</u>

The answer is 2442. Numbering the verses using only nine digits, rather than ten, implies that the poet is actually writing the numbers in a base-9 number system. In this system the number 1000 is typically written 1331 ($= 11^3$); but in our case the possible digits are 0, 2, 3, 4, 5, 6, 7, 8, 9, where 2 stands for 1, 3 for 2, 4 for 3, and so on. The last verse of the poem will hence be assigned the number 2442.

GE3C.

There are 4 distinct points on the plane K, L, M, N, such that the midpoint of MN does not coincide with the midpoint of KL. Consider a rectangle whose sides (or straight lines containing the sides) go through each of the assigned points; more precisely, we assume that M and N are on opposite sides of the rectangle, and that K and L are on the other two opposite sides. Consider the locus of the centers of the rectangles that can be constructed in this way (the center of a rectangle is the point where the two diagonals intersect). Such locus is

A) a circle

B) a straight line

C) a pair of parallel lines

D) a pair of intersecting lines

E) two adjacent segments

<u>Solution</u>

The answer is A). Let P be the midpoint of MN and let Q be the midpoint of KL. Point P is on a line parallel to two opposite sides of the rectangle and it has the same distance from each of them. Analogous considerations can be made for point Q. Let O be the center of the rectangle; the angle $\angle POQ$ is right. Therefore the desired locus is the circle with diameter PQ (in limit cases, O can coincide with P or with Q).