Mediterranean Youth Mathematical Championship (MYMC)
Rome, July 18, 2013

Morning round

WE1

(Filippo Calandri, *Aritmetica*, 1491)

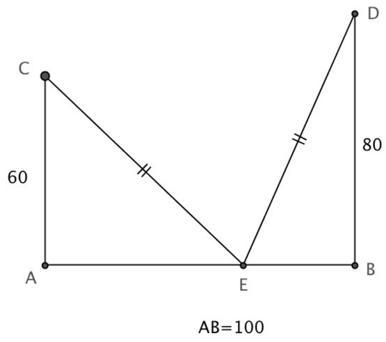
There are two towers: one is *60* meters tall and the other is *80* meters tall; between one tower and the other there is a distance of *100* meters. On the ground there is a water fountain, found between the bases of the two towers. It is known that two birds, leaving simultaneously from the tops of the two towers and flying with the same speed, will reach the fountain at the same moment. What is the distance between the fountain and the shortest tower?

Copy of the page of the 1491 book:

Sono dua torre inuno piano che
luna e alta 60 bra lautra e al
ta 80 bra et daluna torre alaltra
e 100 bra : et intra queste dua to
rre e una fonte daqua intalluo
gho che movendosi dua ucelli disu
dette torre et volando di pari volo
giungono anuotta adetta fonte : Vo
sapera quato ladetta fonte sara pre
sso aciascuna torre : facosi prima
multipricha laltezza delle torre
ognuna per se cioe 80 vie 80 fa 6400
poi 60 vie 60 fa 3600 tralo di 6400 re
sta 2800 poi multipricha ladistantia

Solution

The situation is represented by the following figure.



$AE=x$

$60^2+x^2=80^2+(100-x)^2$

$x= 64.$

Note: the point E can be found with ruler and compass, constructing the perpendicular bisector of the segment CD.

WE2

(Leonhard Euler, *Anleitung zur Algebra*, 1770)

Find two positive numbers, one the double of the other, such that if their product is added to their sum, the result is 90.

Solution

Let x and y be the two numbers, $y=2x$. We get the equation $2x^2+3x=90$. The only positive solution is $x=6$ (so that $y=12$).

WE3

(Rafael Bombelli, *L'algebra*, 1572)

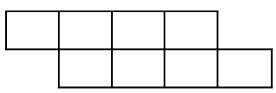
Find three integers greater than 100 such that the product of any two of the three numbers, increased by 1, is a square number.

Solution

One solution consists in the observation that the numbers $a-1$, $a+1$, $4a$ satisfy the statement (for every a).

WE4

How many ways can we fill the cells of the following table with the colors blue, red, and green



such that no two adjacent cells (horizontally or vertically) contain the same color?

Solution

The answer is in $216=54\cdot 2\cdot 2$ ways, for there are 54 ways of filling the following rectangular table:

The rectangular table can be filled in 54 ways because we can choose $3\cdot 2=6$ fillings of the two top leftmost cells, and for every choice we have 9 ways of filling the whole table. We illustrate the case

B	R	

the others being similar (B, R, G stand for blue, red, green, respectively).
If we add G in the first row, then another G under B determines

B	R	G
G	B	R

but putting R under B we have the following three cases

B	R	G
R	B	R

B	R	G
R	G	B

B	R	G
R	G	R

If we add B in the first row, then putting G in the leftmost cell of the second row, we have two cases

B	R	B
G	B	R

B	R	B
G	B	G

but putting R in the leftmost cell of the second row, we have three cases

B	R	B
R	G	R

B	R	B
R	B	R

B	R	B
R	B	G

WE5

We are given two quadrilaterals *ABCD* and *EFGH*, each having all internal angles smaller than $180°$, and with angle bisectors (half lines that divide the internal angles into two equal parts) which pairwise intersect at internal points of the quadrilateral. We know that, for the first quadrilateral, the measures of the internal angles at A and B are α and β, while for the second quadrilateral α and β are the measures of the internal angles at E and G (i.e. non-consecutive vertices). Let P be the intersection point of the bisectors of the two angles at C and D, and θ the convex angle *CPD*; furthermore, let Q be the intersection point of the bisectors of the two angles at F and H, and φ the convex angle *FQH*. Then:

A) $\theta = \varphi$ if and only if α and β are supplementary
B) $\theta = \varphi$ if and only if α and β are complementary
C) θ is always different from φ
D) θ is always equal to φ
E) $\theta = \varphi$ if and only if $\alpha = \beta$

Solution

The answer is C).
For the first quadrilateral, the sum of the internal angles at C and D is $360° - (\alpha + \beta)$ and we obtain $\theta = 180° - (360° - (\alpha + \beta))/2 = (\alpha + \beta)/2$. Let us now suppose, without loss of generality, that $\alpha \geq \beta$; for the quadrilateral *FGHQ* the angle at Q is $360° - \varphi$ and we have $\varphi = \beta + (360° - (\alpha + \beta))/2$ and therefore $\theta = \varphi$ for $(\alpha + \beta)/2 = (\beta - \alpha)/2 + 180°$ i.e. only when $\alpha = 180°$. But this contradicts our initial hypotheses.

WE6

In the Cartesian plane the equation $ax^2 - |ay^2| + ab = 0$ (where a, b are real numbers) defines a circle with its center at the origin and radius greater than 0 if and only if

A) $a < 0$ and $b < 0$
B) $a \neq 0$ and $b < 0$
C) $a < 0$ and $b > 0$
D) $ab < 0$
E) $ab > 0$

<u>Solution</u>

The answer is A). For the coefficients of x^2 and y^2 to be equal, a must be negative. The square of the radius is positive if and only if b is also negative.

WE7

Consider the following polyhedrons in the geometry of space: a cube, a convex prism with 8-sided bases, a convex pyramid with an 8-sided base (the bases of prism and pyramid are regular 8-gons). All of the numbers written in the following table are correct, except for one; furthermore, one of the boxes is empty. Write the product of the missing number with the number that should substitute the incorrect number.

Polyhedron	Maximum number of edges of the polyhedron which lie on pairwise skew lines (recall that two lines are said to be "skew" if no plane contains both of them)	Maximum number of sides of a polygon obtained as a plane section of the polyhedron
cube	*3*	*6*
prism with *8*-sided bases	*3*	
pyramid with *8*-sided base	*3*	*9*

<u>Solution</u>

Let *ABCD* and *A'B'C'D'* be two opposite faces of the cube, with *AA'*, *BB'*, *CC'*, *DD'* edges that are perpendicular to the bases.

Then the lines *AB*, *CC'*, *D'A'* are pairwise skew. On the other hand, each edge of the cube is parallel to three other edges, and therefore, however we choose four lines, two of them will be parallel.

There exists a plane which intersects all the faces of the cube: for example the plane passing through all of the midpoints of the edges *AB*, *BC*, *CC'*, *C'D'*, *D'A'*, *A'A* (in this case, the plane section is a regular hexagon).

In a prism, we can find at most three edges which lie on skew lines: one in each base and one lateral edge.

The answer in the empty box is *10*: it comes from a plane which intersects all of the faces of the prism. For example, we can consider a plane which intersects all of the lateral edges and we can incline it in such a way as to also intersect the two bases.

In a pyramid, we can find at most *2* skew lines (the number in the table is incorrect): one edge from the base and one lateral edge, because every base edge is coplanar with the first edge chosen, and every lateral edge is coplanar with the second edge chosen.

The number *9* is correct: the situation is analogous with that of the prism.

The answer is *20*.

WE8

We have 2013 points (numbered in order from 1 to 2013) which are the vertices of a regular 2013-agon. These vertices are divided into three groups: those from 1 to 671, those from 672 to 1342, and those from 1343 to 2013. What is the smallest number n such that, by choosing n points from each group, we can be certain that the points chosen include the vertices of an equilateral triangle?

<u>Solution</u>

The answer is 448.

Firstly, we note that each point a is the vertex of one and only one equilateral triangle, namely the triangle whose vertices are the points a, $a+671$, $a+1342$ (where addition is performed modulo 2013). Therefore, with the given points we can obtain a total of 671 equilateral triangles. If we choose 448 points from each group, we discard $671 - 448 = 223$ points from each group. Therefore, our choice of points in the first group leads us to discard 223 equilateral triangles, and in addition we will discard – at most – another 223 equilateral triangles for every choice of 448 points in either of the remaining two groups. But as $223\times3 = 669$, we are sure to have at least one remaining equilateral triangle (rather, we will have at least 2).

Finally, we show that $n = 447$ does not guarantee the presence of an equilateral triangle. In fact, we do not obtain any equilateral triangles by choosing the points from 1 to 447 in the first group, the points from 896 to 1342 in the second group, and finally, in the third group, the points from 1343 to 1566 alongside the points from 1791 to 2013.

WE9

Let n be an odd integer between 90 and 100. How many n are such that $4/n$ is the sum of the reciprocals of two positive integers?

- A) 1
- B) 2
- C) 3
- D) 4
- E) 5

<u>Solution</u>

The answer is D).

In fact, $4/91=1/35+1/65$, $4/93=1/93+1/31$, $4/95=1/30+1/114$ and $4/99=1/45+1/55$.

Now, if $4/97$ were equal to $1/a + 1/b$, with a and b positive integers, $4ab=97(a+b)$ would require that a or b be divisible by the prime number 97: let's say that $a=97c$.

It would follow that $4cb=97c+b$, from which $c=b/(4b-97)$. But $c\geq1$ would require that $b\geq(4b-97)$, and so b would have to satisfy $b\leq32$.

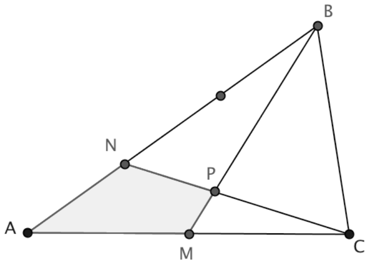
Furthermore, positive c would require that $4b-97$ be positive, meaning that the integer b would also have to satisfy $b\geq25$.

For each of the integer values of b between 25 and 32, $c=b/(4b-97)$ is never an integer: we arrive at a contradiction.

In general, we can prove that $4/n$ can be expressed in the desired form if and only if n has a prime factor congruent to 3 modulo 4.

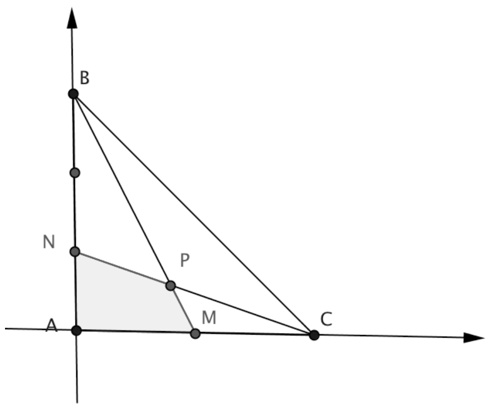
WE10

Given a triangle *ABC*, let *N* be a point on the side *AB* such that *AN* is one third of *AB*, and let *M* be a point on the side *AC* such that *AM* is half of *AC*. Connecting *N* to *C* and *M* to *B* with two segments, we then define *P* as the intersection point of these two segments. Calculate the fraction expressing the ratio between the area of the triangle *ABC* and that of the quadrilateral *ANPM*.



First solution

The problem is affine, in the sense that the requested ratio is preserved by any affine transformation. So we can take the triangle we prefer, for instance a right isosceles triangle. Thus we assume that the angle at *A* is a right angle, and choose an appropriate Cartesian coordinate system, such as the one in the figure.



$A=(0,0)$, $B=(0,1)$, $C=(1,0)$, $N=(0,1/3)$, $M=(1/2,0)$

NC : $x+3y=1$

MB : $2x+y=1$

$P=(2/5,1/5)$

Area $ABM = 1/4$

Area $BNP = \frac{1}{2}\frac{2}{3}\frac{2}{5} = \frac{2}{15}$

Area $ANPM = \frac{1}{4} - \frac{2}{15} = \frac{7}{60}$

The requested ratio is *30/7*.

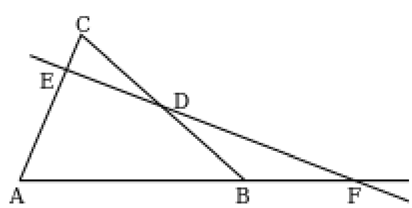
Second solution

Draw the segment *AP* and consider the areas of the five triangles in which *ABC* is decomposed. Let us set: *area AMP = area MCP = a* (these two triangles have congruent bases and the same height), *area BPC = b*, *area APN = c* ; then *area NPB = 2c* (the basis of the latter triangle is twice the basis of *APN*).

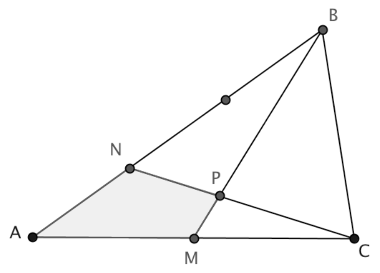
With some argument similar to the previous ones, we get: $a+c+2c = a+b$ and $2c+b = 2(c+2a)$. We deduce $b = 3c$ and $b = 4a$. It follows that *area ABC* $= 2a+b+3c = 10a$, while *area AMPN* $= a+c = (7/3)a$. The ratio is *30/7*.

Third solution

Let us recall Menelaus' Theorem in plane geometry. With reference to the figure



the following equality holds: *AF·BD·CE* = – *FB·DC·EA*. This equation uses signed lengths of segments, in other words the length *AB* is taken to be positive or negative according to whether *A* is to the left or right of *B* in some fixed orientation of the line.



We apply Menelaus' Theorem to the triangle *ABM*, which is divided by the segment *NC*.
We have (the ratios are in absolute value)

$$\frac{AN}{NB} \times \frac{BP}{PM} \times \frac{MC}{AC} = 1$$

which gives us *BP:PM=4*, and therefore the height of the triangle *MPC* is *1/5* that of the triangle *ABC*, and the base is half that of the triangle *ABC*.
The area of the triangle *MPC* is therefore *1/10* that of the triangle *ABC*. We observe that the area of the quadrilateral *ANPM* is given by the difference between the area of the triangle *ANC* and that of the triangle *MPC*, which we have just calculated.
But the base *AN* of the triangle *ANC* is *1/3* of the base *AB* of the triangle *ABC*, while the heights are the same. Therefore, indicating with *S* the area of the triangle *ABC*, we have:

$$\text{Area ANPM} = \frac{S}{3} - \frac{S}{10} = \frac{7}{30} S$$