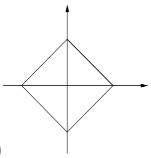
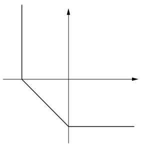
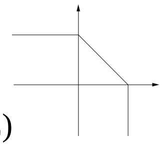
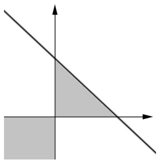
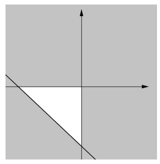
Mediterranean Youth Mathematical Championship (MYMC)
Trieste, July 8, 2015

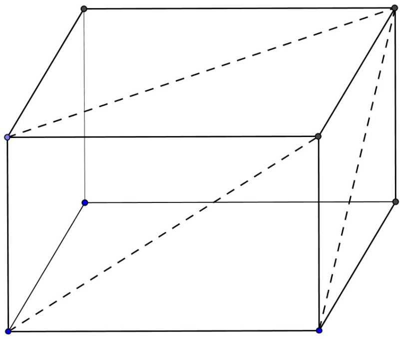
Afternoon round – Last stage

RE2A
Which of the five following subsets of the Cartesian plane corresponds to the set of pairs (x, y) of real numbers satisfying $|x| + |y| = x + y + 2$? [where $|z|$ is the absolute value of z]

A)  B)  C)  D)  E) 

Solution
Using the definition of absolute value, it follows from $|x| + |y| = x + y + 2$ that:
$x + y = x + y + 2$, therefore $0 = 2$ if both x and y are non-negative;
$x - y = x + y + 2$, therefore $y = -1$ if x is positive and y is negative;
$-x + y = x + y + 2$, therefore $x = -1$ if x is negative and y is positive;
$-x - y = x + y + 2$, therefore $x + y + 1 = 0$ if both x and y are non-positive.
Therefore, the correct representation is B).

RE2B
A box in the shape of a parallelepiped has 6 rectangular faces. The three diagonals illustrated in the figure have lengths 17, 25, and $4\sqrt{29}$. What is the volume of the box?



Solution
Let x, y, and z be the three dimensions of the parallelepiped; we have the following system of equations:
$x^2 + y^2 = 289$
$x^2 + z^2 = 625$
$z^2 + y^2 = 464$
We obtain the following solutions: $x^2 = 225$, $y^2 = 64$, $z^2 = 400$, giving us $x = 15$, $y = 8$, $z = 20$.
The volume is therefore 2400.

A point of interest: In the text, one length is expressed with a square root. Is it possible to find a parallelepiped with three integer diagonals, whose dimensions are also integers?
The answer is yes: The smallest solution is obtained when the three diagonals have lengths 125, 267, and 244; in this case, the dimensions are $x = 44$, $y = 117$, and $z = 240$. These parallelepipeds are called ‘Euler bricks’. It is not known whether there exists some Euler brick whose internal diagonal is also an integer (i.e. the diagonal which connects two opposite vertices).

GE2A

An ancient Mediterranean civilisation had a system for counting which used only the following three symbols:
I (for the number 1),
Q (for the number 4),
and
D (for the number 12).
Numbers were written as set combination of these symbols, from the largest symbol to the smallest (for example, 21 was written DQQI), using the least possible number of symbols.
How many ‘Q’ symbols were needed to write all of the numbers from 1 to 100?

Solution

The answer is 97.
In order to minimize the number of symbols needed to represent a number n, it is best to use Ds wherever possible, which would be $[n/12]$ times (where $[x]$ is the integer part of x). Therefore the problem can be reduced to counting the number of Qs needed for the numbers from 1 to 12; we then multiply this number by 8 and finally add the Qs needed to represent the numbers from 97 to 100. It is easy to see that 1 = I, 2 = II, 3 = III, 4 = Q, 5 = QI, 6 = QII, 7 = QIII, 8 = QQ, 9 = QQI, 10 = QQII, 11 = QQIII, and 12 = D, using a total of 12 Qs. Among the last 4 numbers, only 100 needs one Q, therefore we need 97 Qs in total.

GE2B

In binary, the number 2015 is written as 11111011111; this is a palindromic number, meaning that the sequence of digits remains the same when reading the number from left to right or right to left. After 2015, what is the next year that is a palindromic number when written in binary? Write the year in the usual base 10 notation.

Solution

The answer is 2047.
If we modify one of the first five digits of the number (i.e. “11111”), we will obtain a smaller number. Therefore the only number which is palindromic in binary, which has 11 digits and is greater than the given number is $11111111111 = 2^{11} - 1 = 2047$. Obviously there are palindromic numbers with a greater number of digits: in particular, $100000000001 = 2^{11} + 1 = 2049$ is palindromic.