Mediterranean Youth Mathematical Championship (MYMC)
Rome, July 18, 2013

<u>Afternoon round – Third stage</u>

RE3A

Let f be a function from the set of positive integers into the set of integers. It is known that:
$f(1) = 0$
if $n>0$, then $f(2n) = 2f(n)$
if $n>0$, then $f(2n+1) = 2f(n)+1$.
Find $f(2013)$.

Solution

The answer is *989*. We can find the answer by computing the values of $f(n)$ inductively (for $n = 1, 3, 7, 15, 31, 62, 125, ...$). But it is not too hard to prove that $f(n)$ is the number which is obtained by writing n in base *2* and "deleting" the first *1* on the left.

RE3B

In an ancient Mediterranean country, the currency consisted of coins of *2* and *5* "mymc", and obviously certain sums of money could be obtained in various different ways: for example, for a payment of *12* mymc one could use either six coins of *2* mymc, or one coin of *2* mymc and two coins of *5*. What is the maximum sum of money that could be obtained with seven different possible combinations of coins?

Solution

Let b be the desired sum.
If b is even, then b can be obtained with coins of *2* mymc without any coins of *5* mymc. The other possibilities can be found by substituting five *2* mymc coins with two *5* mymc coins. This process can be done $[b/10]$ times, because each time we change *10* mymc. (*H*ere $[x]$ stands for the largest integer a such that $a \le x$.) In total, there are $[b/10]+1$ ways of obtaining b.
If b is odd, then in any possible combination, there must be a *5* mymc coin; seeing as $b–5$ is even, we return to the previous situation: therefore, as we have seen, there are $[(b–5)/10] + 1$ ways of obtaining b.
If $[(b–5)/10] + 1 = 7$, then $60 \le b-5 < 70$. The answer is $b = 73$.

GE3A

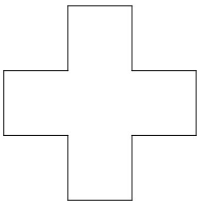
What is the minimum value of the product $2^a \cdot 3^b$ that can be obtained with a and b non-negative integers, and such that both conditions $a+2b \ge 9$ and $2a+7b \ge 21$ are satisfied?

Solution

The minimum value is *162*.
Take a Cartesian plane with horizontal a – axis and vertical b – axis. The pairs (a,b) of integers, which are acceptable as exponents in $2^a \cdot 3^b$, all lie in an unbounded region of the first quadrant, identified by means of the axes and of the lines $a+2b = 9$ e $2a+7b = 21$. The points of the region that are likely to minimize $2^a \cdot 3^b$ are located on those two lines or nearby. They are: $(11,0), (7,1), (5,2), (3,3), (1,4), (0,5)$. Calculating the value of $2^a \cdot 3^b$ at each of the six points, we see that the minimum value is $2^1 \cdot 3^4 = 162$.

GE3B

In a given Cartesian plane, we consider any Greek cross of unit sides, whose vertices are points with integer coordinates.



We want to enclose this cross within a convex polygon, whose vertices are also points with integer coordinates and such that no side of the polygon touches or intersects with the cross.
What can be said about the minimum area M of a polygon that satisfies these conditions?

A) $M \leq 12$
B) $12 < M \leq 13$
C) $13 < M \leq 14$
D) $14 < M \leq 15$
E) $15 < M$

Solution

The correct answer is $M=13$, therefore B).
If $(0,1)$, $(0,2)$, $(1,2)$, $(1,3)$, $(2,3)$, $(2,2)$, $(3,2)$, $(3,1)$, $(2,1)$, $(2,0)$, $(1,0)$, $(1,1)$ are the vertices of the cross on the Cartesian plane given, $(-1,1)$, $(1,4)$, $(4,2)$, $(2,-1)$ are the vertices of a polygon of minimum area. Such a polygon (being a square of sides $\sqrt{(9+4)} = \sqrt{(13)}$) has area 13 because it is made up of a central square of area 1 with four triangles of base 3 and height 2 surrounding it.
That the minimum area is 13 can also be proven using the so-called Pick's formula. Given a polygon P having as vertices points with integer coordinates, Pick's formula expresses the area A of P in terms of the number i of lattice points in the interior of P and the number b of lattice points on the perimeter of P:
$A = i + b/2 - 1$.