Mediterranean Youth Mathematical Championship (MYMC)
Rome, July 18, 2013

Afternoon round – First stage

RE1A

Writing the positive integer n to the left of the number 2013, namely attaching the digits of the number n to the left side of the digits of 2013, let us count for how many different n the new number obtained is a multiple of n (for example, if $n=1$ the new number is 12013 which is a multiple of 1, while if $n=2$ the new number is 22013 which is not a multiple of 2, and so on).
Repeat the same count with the number 2014 and then 2015. Regarding the results of the counts obtained in the three cases, we can say that:

- A) They are pairwise different
- B) They are identical
- C) The first is equal to the second, but the third is different
- D) The first is equal to the third, but the second is different
- E) The second is equal to the third, but the first is different

Solution

The answer is B).
For the number 2013, the property to be verified is that $n2013 = k\cdot n$ for some integer k, namely that $10000\cdot n + 2013 = k\cdot n$. We get $k = 10000 + 2013/n$ which is an integer if n divides 2013.
Now $2013 = 3\cdot 11\cdot 61$ and so there are 8 divisors of 2013 (specifically $1, 3, 11, 33, 61, 183, 671, 2013$).
The result of the count for 2013 is 8.
But $2014 = 2\cdot 19\cdot 53$ and $2015 = 5\cdot 13\cdot 31$ have the same number of prime factors as 2013 and with the same multiplicity, therefore they both have the same number of divisors as 2013. So the results of the counts for 2014 and 2015 are also 8 and are therefore identical to that for 2013.

RE1B

We fix 15 points on a circle. We consider all convex angles of the type ABC, where A, B, C are three distinct points of the 15 points given. What is the maximum number of angles of different sizes that can be formed in this way? (counting two or more congruent angles as just one size)

Solution

Let n be the number of points on the circle. If A and C are two non-adjacent points (non-consecutive), then there are two angle sizes of the type ABC, because there are two arcs with endpoints A, C along the circle. The number of pairs of non-adjacent points is $n(n–3)/2$; as a result, there are at most $n(n–3)$ different sizes (if the distances between the pairs of points are pairwise different).
If A and C are two adjacent points, then all angles of the type ABC are of the same size.
In total, the possible sizes are at most $n(n-3) + n = n^2-2n$.
For $n = 15$, the answer is 195.

GE1A

(Mathematical folklore)

A Raja dies, leaving to his numerous daughters a certain number of pearls, requesting that they be shared out in the following way: the eldest would receive one pearl, and a seventh of those then remaining. The second daughter would then receive two pearls, and a seventh of those then remaining, the third would then receive three pearls, and again a seventh of those then remaining, and so on. The youngest daughter ran to the judge complaining that this complex system was hugely unfair. The judge who, as was tradition, was skilled in problem-solving, immediately replied that the girl was mistaken: the proposed division was correct, the pearls would be equally divided and each of the daughters would receive the same number of pearls. How many pearls were there? How many daughters did the Raja have?

Solution

x is the number of pearls owned by the Raja. The first daughter receives $1+\frac{1}{7}(x-1)$ pearls and $x-[1+\frac{1}{7}(x-1)]=\frac{6}{7}(x-1)$ pearls remain. The second daughter receives $2+\frac{1}{7}\left[\frac{6}{7}(x-1)-2\right]$. As both daughters receive the same number of pearls, we obtain the equation:

$$2+\frac{1}{7}\left[\frac{6}{7}(x-1)-2\right]=1+\frac{1}{7}(x-1)$$

with solution x = *36*. The first daughter therefore inherits *1+35/7=6* pearls, meaning *30* pearls remain. The second daughter inherits *2+28/7=6* pearls, so *24* remain. The third daughter inherits *3+21/7=6* pearls, so *18* pearls remain. Each time we remove *6* pearls. When we reach the sixth daughter, all of the *36* pearls have been shared out. Therefore the answer is: *36* pearls and *6* daughters.

Alternative solution

If n is the number of daughters, then the nth daughter will be given n pearls, because there will be no remaining pearls as she is the last daughter and all of the pearls will therefore have been shared out. Therefore the number n of pearls that each daughter inherits must be equal to the number of daughters. The total number of pearls will therefore be n^2 given that every time, n times, n pearls are removed and at the end no pearls remain. To find the number n we observe that the first daughter inherits $1+\frac{1}{7}(n^2-1)=n$ pearls. This equation can be reduced to the form $n^2-7n+6=0$ which has solutions $n=1$ and $n=6$. As the Raja has more than one daughter, $n=6$.

GE1B

The inhabitants of a distant island are of two kinds: the knaves, who always lie, and the knights, who always tell the truth.

There are n inhabitants H_1, H_2, ..., H_n on the island, arranged in a circle; when speaking of their "successor", H_1 is referring to H_2, H_2 is referring to H_3, ..., H_n is referring to H_1. Of the n inhabitants, k say:

"I am the same kind as my successor,"

while the remaining $n–k$ say:

"I am a different kind to my successor."
Can we determine who are the knaves and who are the knights?

A) For any n and k (with $k \leq n$), we can determine in one and only one way the kind of each H_i .
B) For any n and k (with $k \leq n$), we can determine in one and only one way the total number of knights and of knaves, but, in general, we cannot determine the kind of each H_i .
C) For any n and k (with $k \leq n$), we can determine in at least one way the total number of knights and of knaves, but, in general, there are more possible ways.
D) For any n and k (with $k \leq n$), if $n + k$ is odd, then there is a contradiction, i.e. no way exists to determine the kind of each H_i.
E) For any n and k (with $k \leq n$), if $n + k$ is even, then there is a contradiction, i.e. no way exists to determine the kind of each H_i.

Solution

The correct answer is the first. The key observation is the following. If H_i says "I am the same kind as my successor", then H_{i+1} is a knight (whether H_i is a knight or a knave); while if H_i says "I am a different kind to my successor", then H_{i+1} must be a knave.
Therefore, the statement of each inhabitant fully determines the kind of its successor, without any restriction.