Mediterranean Youth Mathematical Championship (MYMC)
Rome, July 20, 2017

Morning round

WE1.
Romulus and Remus are shepherds. Each of them owns a number of sheep which is a perfect square (let's say that Romulus has a^2 sheep and Remus has b^2 sheep). We know that the total number of sheep, $x = a^2 + b^2$, lies within the range $97 \leq x \leq 108$. Knowing that

- Romulus has more sheep than Remus;
- each of them has at least two sheep;
- and x is odd,

how many sheep do each of them have?

Solution
The answer is $a^2 = \mathbf{81}$ and $b^2 = \mathbf{16}$. The possible candidates for x are 97, 99, 101, 103, 105, and 107. Because $x = a^2 + b^2$, we can assume that one term is odd and the other is even. If $a = 2c$ and $b = 2d + 1$, then

$$x = 4(c^2 + d^2 + d) + 1.$$

Therefore, the remainder of x when divided by 4 is 1. This rules out $99, 103,$ and 107. Next, observe that $105 = 3 \cdot 5 \cdot 7$. If $105 = a^2 + b^2$, then either both a and b would be divisible by 3 (which is absurd because it would imply that 105 is divisible by 9), or $a = 3c \pm 1$ and $b = 3d \pm 1$. In both cases, x would not be divisible by 3. Finally, a quick calculation shows that the only possibilities are $101 = 1^2 + 10^2$ and $97 = 4^2 + 9^2$. Therefore, Romulus has 81 sheep and Remus has 16 sheep.

WE2.
The area S of an octagon inscribed in a circle with four consecutive sides of length 20 and four consecutive sides of length 17 lies in the interval

A) $1430 \leq S < 1480$
B) $1480 \leq S < 1530$
C) $1530 \leq S < 1580$
D) $1580 \leq S < 1630$
E) $1630 \leq S < 1680$

Solution
The answer is **E)**. The sides can actually be drawn in any order, considering that the answer depends on the sum of the areas of the isosceles triangles with bases of length 17 or 20 and their remaining two sides as radii of the circle. If we imagine the triangles to be alternate (one with base 20 and the next with base 17, and so on), we find that the octagon can be obtained from a square of sides $20 + 2x$ by

cutting off right-angled isosceles triangles with legs of length x and hypotenuse of length 17. The area S is therefore $S = (20 + 2x)^2 - 2x^2$ with $\sqrt{2}x = 17$. Therefore, $S = 689 + 680\sqrt{2}$, and considering that $1.41 < \sqrt{2} < 1.42$ we find that $1647.8 < S < 1654.6$.

WE3. (Leonardo Pisano, *Liber Abbaci*, 1202)

A certain man buys 12 birds, which are partridges, pigeons, and sparrows [three different kinds of bird], for 12 denari. A partridge is worth 2 denari, and two pigeons are sold for 1 denaro, and four sparrows for 1 denaro. It is sought how many birds he buys of each kind [at least one bird of every kind].

Solution

The answer is **5** partridges, **1** pigeon, and **6** sparrows. For if x, y, and z are the (positive) numbers of partridges, pigeons and sparrows, respectively, then we have $2x + \frac{1}{2}y + \frac{1}{4}z = 12$ (denari) and $x + y + z = 12$ (birds) — that is, $8x + 2y + z = 48$ and $x + y + z = 12$, which is equivalent to $7x + y = 36$ and $x + y + z = 12$. Because none of the unknowns can exceed 12, the only possible solutions of $7x + y = 36$ are $x = 5, y = 1$ or $x = 4, y = 8$. But $x = 4, y = 8$ implies that $z = 0$, which is unacceptable. Hence the only solution is $x = 5, y = 1, z = 6$.

WE4.

How many pairs of natural numbers (m, n) exist such that $m^2 - n^2 = 800$?

A) One

B) Three

C) Six

D) Ten

E) Infinite

Solution

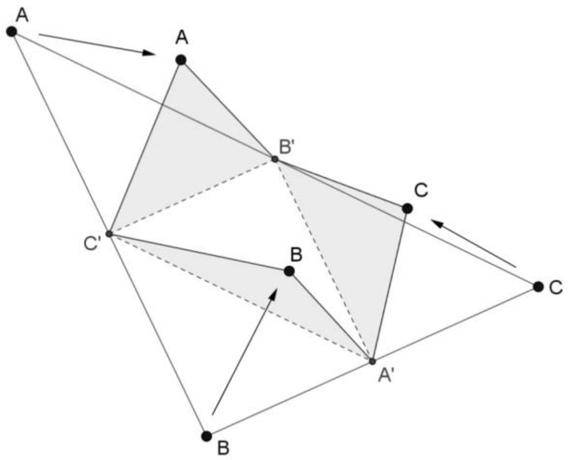
The answer is **C)**. Because $m^2 - n^2 = (m + n)(m - n)$, we need to find two natural numbers whose product is 800. The two numbers $m + n$ and $m - n$ must have the same parity, and so we need to decompose 800 into two even factors. This can be done in six different ways:

$$800 = 2 \cdot 400 = 4 \cdot 200 = 8 \cdot 100 = 10 \cdot 80 = 16 \cdot 50 = 20 \cdot 40.$$

In the first case, we obtain the system $\begin{cases} m + n = 400 \\ m - n = 2 \end{cases}$ whose solution is $m = 201, n = 199$. Proceeding in the same way with the other factor pairs, we obtain (102, 98), (54, 46), (45, 35), (33, 17), and (30, 10).

WE5.

An isosceles triangle ABC has sides of lengths $AB = AC = 40$ and $BC = 48$. Let A', B', and C' be the midpoints of the sides BC, CA, and AB, respectively. As shown in the figure, we fold the triangle along the segments $A'B'$, $B'C'$, and $C'A'$ such that the three vertices A, B, and C meet at the point V (not shown in the figure).



(The figure is not drawn proportionally with respect to the lengths of the sides.)

Let VH be the height of the pyramid with vertex Vwhich has $A'B'C'$ as its base. The length of VH is

A) between 15 and 16

B) between 16 and17

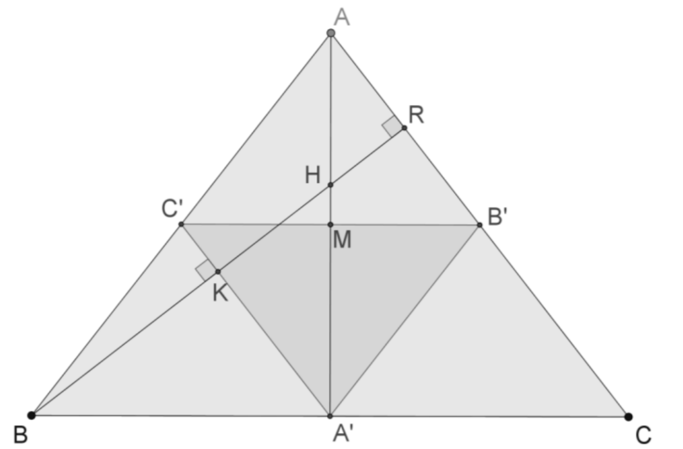
C) between 17 and18

D) between 18 and19

E) between 19 and20

Solution

The answer is **A)**.



As in the figure, let BR be the height of ABC relative to AC, and let K be the intersection of BR with $A'C'$. When the triangle $BA'C'$ rotates around $A'C'$, the vertex B remains on the plane passing through $K$and perpendicular to $A'C'$ and thus to AC. Owing to symmetry, the vertex V lies on the plane passing through AA' perpendicular to ABC. Therefore, the projection of the vertex Von the plane ABC is the orthocentre $H$of the triangle ABC. We find that $AA' = 32$, from which it follows that $AM = A'M = 16$. Furthermore, the triangles $AA'C$, BCR, and AHR are right-angled triangles which are pairwise similar: in each triangle the sides are proportional to the triplet $(3, 4, 5)$. It follows that $CR = 28.8$, $AR = 11.2$,

$AH = 14$, and $HM = 2$ (note that the point H lies outside of the triangle $A'B'C'$). The triangle VHM is right angled, with its right angle at H, and $VM = AM$. We conclude that

$$VH^2 = VM^2 - HM^2 = AM^2 - HM^2 = 256 - 4,$$

which means that $VH = 6\sqrt{7} = 15.8745 \ldots$ Note that $15^2 = 225 < VH^2 < 256 = 16^2$ and therefore $15 < VH < 16$.

WE6.

Players A and B play heads or tails with a fair coin. Player A is allowed 20 throws, whereas player B is allowed 21 throws. What is the probability x that player B throws more heads than does player A?

A) $\frac{20}{41} < x < \frac{1}{2}$

B) $x = \frac{1}{2}$

C) $\frac{1}{2} < x < \frac{21}{41}$

D) $x = \frac{21}{41}$

E) $x > \frac{21}{41}$

Solution

The answer is **B)**. Firstly, suppose that each player throws 20 times. The probability p that player A throws more heads than does player B is equal to the probability that player B throws more heads than does player A. If player A throws more heads than does player B, player B's last throw (that is, the 21st) would lead to nothing more than a tie; if, in 20 throws, player B throws more heads than does player A, player B's last throw has no effect. Secondly, if players A and B throw the same number of heads (with a probability of $1 - 2p$), player B has probability $\frac{1}{2}$ of overtaking player A with the last throw. Therefore, the probability that player B throws more heads than does player A is $x = p + \frac{1}{2}(1 - 2p) = \frac{1}{2}$.

An alternative solution is the following: one and only one of the two events "player B throws more heads than does player A" or "player B throws more tails than does player A" can happen. The two events are clearly equally probable, and so the probability of either one occurring is $\frac{1}{2}$.

WE7.

A man walks his dog on a straight path; both man and dog walk at a constant speed, but the dog's speed is faster than the man's. The man and the dog leave together, both travelling in the same direction. The dog, which is faster, reaches a distance of 100 metres away from the man. At this point, the dog turns around and returns to its owner; we know that by the time the dog reaches the man, the man has walked 100 metres from when they left. The dog repeats this same routine throughout the whole walk, while the man always keeps walking in the same direction. Knowing that the total distance walked by the man is 2 kilometres, what is the total distance travelled by the dog?

A) 4000 m

B) $2000 \cdot (1 + \sqrt{2})$m

C) $2000 \cdot (2-\sqrt{2})$m

D) 3000 m

E) $2000 \cdot \left(1+\frac{\sqrt{2}}{2}\right)$ m

Solution

The answer is **B)**. Let v be the speed of the man and V the speed of the dog. Let t be the time it takes the man to walk 100 metres. Then we know that $v \cdot t = 100$ m.

The dog travels ahead of the man (in the same direction) for a time t_s to then return to the man in the time $t - t_s$. During the time t_s, the man and the dog move away from each other at the speed $V - v$; at the start of this time, the distance between them is 0 metres, whereas at the end it is 100 metres. Therefore,

$$V - v = \frac{100 \text{ m}}{t_s}.$$

During the time $t - t_s$, the man and the dog move towards one another at the speed $V + v$; at the start of this time, the distance between them is 100 metres, whereas at the end it is 0 metres. Therefore,

$$V + v = \frac{100 \text{ m}}{t - t_s}.$$

Cancelling out the t and $t_s$in these three equations,

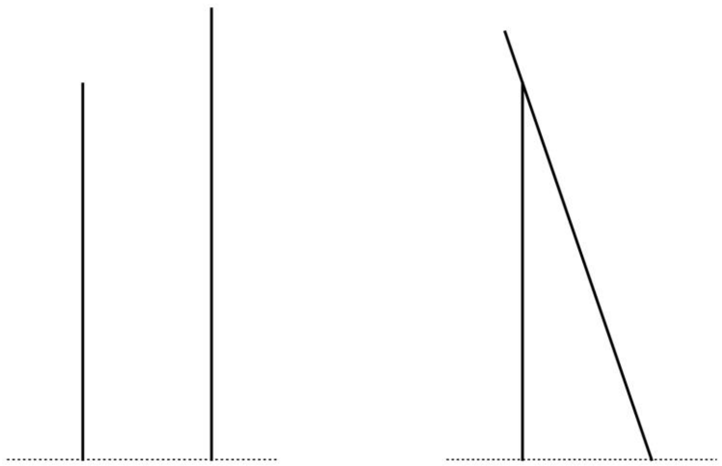
$$v \cdot t = 100 \text{ m}, \quad V - v = \frac{100 \text{ m}}{t_s}, \quad \text{and} \quad V + v = \frac{100 \text{ m}}{t - t_s},$$

with simple algebraic steps, we find that $V^2 - 2Vv - v^2 = 0$.

We are interested in the ratio r between the speed of the dog and the speed of the man, namely $r = V/v$. Dividing both sides of our speed equation by v^2 and substituting r, we find that $r^2 - 2r - 1 = 0$. Excluding the negative solution, we find that $r = 1 + \sqrt{2}$. Because the man walks a total distance of 2 kilometres, the dog travels a total of $(1 + \sqrt{2}) \cdot 2000$ m.

WE8. (Leonardo Pisano, *Liber Abbaci*, 1202)

On a certain ground are standing two poles that are only 12 feet apart, and the lesser pole is in height 35 feet, and the greater 40 feet; it is sought, if the greater pole will lean on the lesser, then how long will the protruding part be?



Solution

The answer is **3**. By the Pythagorean Theorem, $\sqrt{35^2+12^2}=\sqrt{1225+144}=\sqrt{1369}=37$ is the length of the part which is not protruding, and $40-37=3$.

WE9. (Leonardo Pisano, *Liber Abbaci*, 1202)

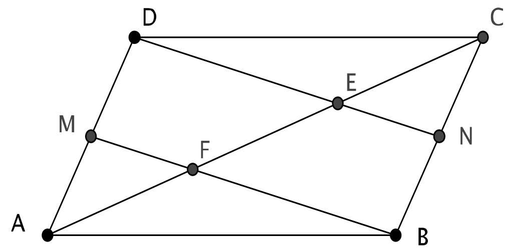
There are two men who propose to go on a long journey [starting together], and one will go 20 miles daily. The other truly goes 1 mile the first day, 2 miles the second, 3 miles the third, and so always one more mile daily to the end [of the journey] when they meet. It is sought for how many days the first is followed [that is, how many days it takes the second man to reach the first man].

Solution

The answer is **39**. If x is the number of days, then $1+2+\ldots+x=20x$, that is $\frac{x(x+1)}{2}=20x$. Because $x\neq 0$, this is equivalent to $x+1=40$, namely, $x=39$.

WE10.

Given a parallelogram with area 24 and vertices A, B, C, D, let M and N be the midpoints of the sides $AD$and BC, as shown in the figure.



The area of the triangle CEN is:

A) 1
B) 2
C) 3
D) 4
E) 5

Solution

The answer is **B)**. In fact, the area of the triangle NCD is a quarter of the area of the parallelogram, namely 6. If E is the intersection point between ND and the diagonal AC, the triangles NCD and NCE have the same height with respect to their bases DN and EN; the ratio of their areas is therefore the same as the ratio of the segments DN and EN.

The segments DN and MB are parallel and of equal length; in particular, EN is parallel to FB. Because N is the midpoint of CB, we find that E is the midpoint of $FC$and that EN is half of FB. But FB is equal to DE, and therefore EN is half of DE and a third of DN. The area of the triangle CEN is therefore a third of the area of NCD.

WE11.

On an island far away lives a community of knights and knaves. The knights always speak the truth, but when the knaves say a phrase which contains a pair of numbers, they always get one (and only one) of these numbers wrong. A tourist visiting the island encounters three islanders and asks them how many of the islanders are male and how many are female.

The first islander replies "m_1 are male and f_1 are female."

The second islander replies "m_2 are male and f_2 are female."

The third islander replies "m_3 are male and f_3 are female."

Which of the following situations could occur and allow the tourist to work out, without uncertainty, the correct number of male and female islanders? (We assume that all numbers m_i and f_i are greater or equal to 3.)

A) $m_1 = m_2 = m_3$ and $f_1 = f_2 = f_3$.

B) $m_1 = m_2 = m_3$, whereas the three numbers f_i are pairwise different to one another.

C) The three numbers m_i are pairwise different to one another, as are the three numbers f_i.

D) $m_1 = m_2$ but $m_3 \neq m_1$; in addition, $f_1 = f_2$ but $f_3 \neq f_1$.

E) $m_1 = m_2$ but $m_3 \neq m_1$; in addition, the three numbers f_i are pairwise different to one another.

F) $m_1 = m_2$ but $m_3 \neq m_1$; in addition, $f_1 = f_2 = f_3$.

Solution

The answer is **E)**. Situation A could occur, but would not permit the tourist to reach any conclusions: the three islanders could be three knaves and, in this case, one of the two numbers would be correct and the other would be incorrect.

In situation B, the three islanders could again be three knaves: this time, the tourist can work out how many islanders are male, but not how many are female.

Situation C could never occur, because at most three of the six numbers, m_i and f_i, can be incorrect, and therefore at least two of the m_i or two of the f_i have to be the same.

Situation D could occur only if the three islanders are knaves; however, the tourist cannot know if the correct numbers are m_1 and f_3 or m_3 and f_1.

Situation E is the only one that could occur and would allow the tourist to work out the correct numbers: the number $m_1 = m_2$ is clearly correct, but also one of the numbers m_3 and f_3 has to be true. Therefore, m_1 are male and $f_3$are female.

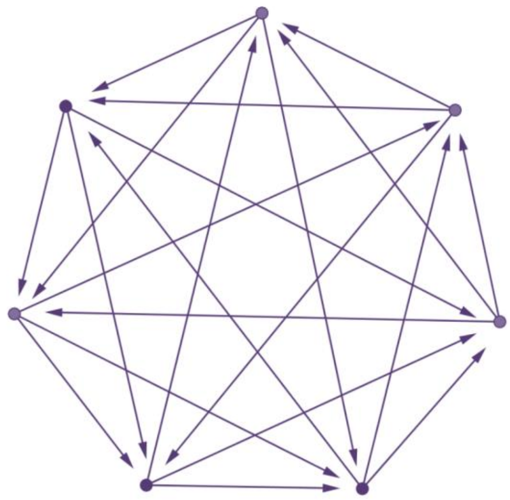
In situation F, the three islanders correctly say how many islanders are female, but the tourist cannot know how many are male.

WE12.

There are n horsemen taking part in a tournament. In this tournament, each horseman battles against every other horseman just once, in a duel in which there can only be one victor (that is, the horsemen cannot tie). At the end of the tournament, it turns out that for every pair of horsemen there is a third horseman who has beaten both of them. What is the minimum value of n for this to be possible?

Solution

The answer is **7**. In the following figure, the seven horsemen are represented by seven points; an arrow which goes from a point A to a point B indicates that, in the duel between those two horsemen, horseman A beat horseman B. Here, our required condition is satisfied: whichever two points we choose, we can see that both points are reached by an arrow with the same origin (that is, the third horseman). We note that, in this scenario, each horseman has won three duels and has lost three duels.



We now prove that the described situation cannot happen with six horsemen. Knowing that for each pair of participants there is a horseman who has beaten both of them, and keeping in mind that with six horsemen there are 15 possible pairings, there must be at least one horseman who has beaten at least three competitors. Suppose that horseman A has beaten horsemen B, C, and D. If we choose a fifth horseman E, we find that the pair of horsemen A and E must have been beaten by the sixth horseman, F. But, in this case, no horsemen could have beaten both A and F. With a similar reasoning, we can show that n cannot be less than 6.

WE13. (A puzzle by Lewis Carroll)

Assume that:

(1) No interesting poems are unpopular among people of real taste.

(2) No modern poetry is free from affectation.

(3) All your poems are on the subject of soap bubbles.

(4) No affected poetry is popular among people of real taste.

(5) No ancient poem is on the subject of soap bubbles.

Which of the following statements can be deduced?

A) Your poetry is not interesting.

B) Modern poetry is on the subject of soap bubbles.

C) Modern poetry is popular among people of real taste.

D) Your poetry is free from affectation.

E) Your poetry is not modern.

Solution

The answer is **A)**. The universe in this puzzle is the collection of all poems, and the five assertions are implications involving the following simple statements:

I = it is interesting;

P = it is popular among people of real taste;

M = it is modern;

A = it is affected;

Y = it is your poem;

S = it is on the subject of soap bubbles.

Denote by → the logical implication—for instance, "Y→ S" means "All your poems are on the subject of soap bubbles". Then let us write each statement symbolically:

(1) I → P; (2) M → A; (3) Y → S; (4) A → not P; (5) not M → not S.

Then we get I → P → not A → not M → not S → not Y.

Thus the solution to the puzzle is I → not Y, or, equivalently, Y → not I. The simplest translation back into words is the (perhaps cruel) statement A).

WE14.

We say a positive integer $n \geq 2$ is separable if we can divide the integers of the set $\{1, 2, \dots, n\}$ into two non-empty disjoint subsets S and P in such a way that the sum of the integers in S equals the product of the integers in P. Among the numbers from 2 to 50, how many of them are separable? [Note that if S or P consists of one element, the sum or product of its only element is the element itself.]

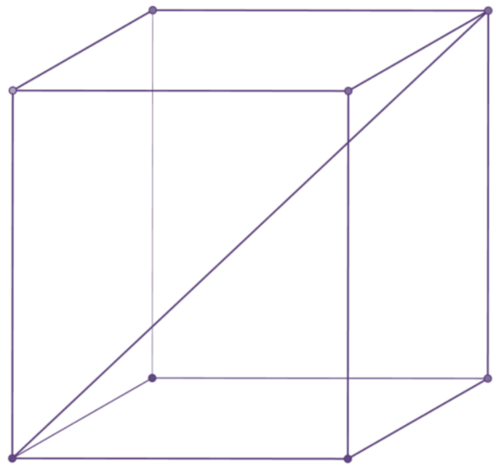
Solution

The answer is **47**. By inspection, 3 is separable, because $S = \{1, 2\}$ and $P = \{3\}$ is a possible bipartition of $\{1, 2, 3\}$ with the desired properties; $n = 2$ and $n = 4$ are not separable. Now, take $n \geq 5$: we can choose a particular set $P = \{1, x, y\}$ such that $S = \{1, 2, \dots, n\} \setminus \{1, x, y\}$.

Consider the sum of all the integers from 1 to n: $1 + 2 + \dots + n = \frac{n(n+1)}{2}$. Then the sum of the integers of S will be $\frac{n(n+1)}{2} - 1 - x - y$. Imposing this condition on a separable n, we must find solutions x and y such that $\frac{n(n+1)}{2} - 1 - x - y = x \cdot y$, which is equivalent to $\frac{n(n+1)}{2} = xy + x + y + 1 = (x + 1)(y + 1)$. If n is even, then let $x = \frac{n}{2} - 1$ and $y = n$; if n is odd, then let $x = \frac{n+1}{2} - 1$ and $y = n - 1$. Therefore, every number larger than or equal to 5 is separable (the fact that $n \geq 5$ means that $1, x$, and y will be three distinct numbers).

WE15.

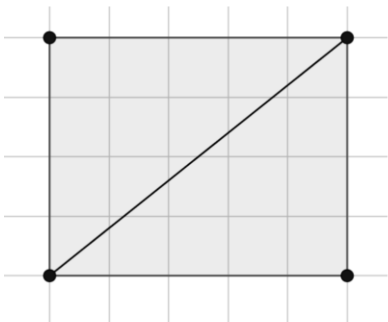
Consider a right parallelepiped of dimensions $5 \times 9 \times 10$ and one of its diagonals, as shown in the figure.



If we divide the parallelepiped into small cubes of side 1, how many of these unit cubes will the diagonal intersect? (Let's say that the diagonal intersects a cube if it passes through a point internal to the cube.)

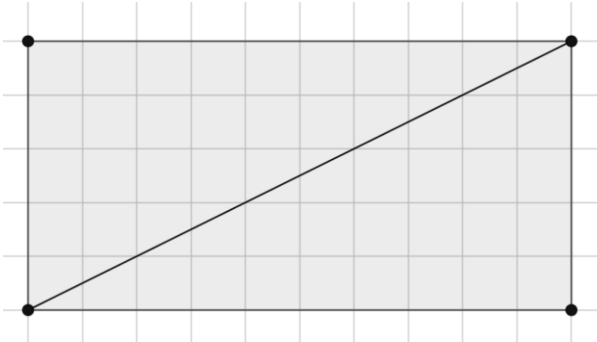
Solution

The answer is **18**. It's useful to address the same problem in two dimensions; for example, consider a rectangle of size 4×5.



We draw straight lines to divide our rectangle into squares of side 1: we draw three lines parallel to the base ($3 = 4 - 1$) and four lines parallel to the height ($4 = 5 - 1$). Each of these seven lines meets the diagonal at a different point: these seven points divide the diagonal into eight segments. This means that the diagonal intersects eight unit squares. Note that $8 = (4 - 1) + (5 - 1) + 1$.

If, however, the rectangle's sides are not co-prime, some of the points of intersection of the division lines and the diagonal will coincide. In the case of a rectangle of size 4×6 (left-hand figure), the diagonal is divided into eight segments rather than $3 + 5 + 1 = 9$ (because two lines meet the diagonal at the same point). In the case of a rectangle of size 5×10 (right-hand figure), the diagonal is divided into only ten segments: the nine lines parallel to the height divide the diagonal into ten parts and the four lines parallel to the base do not create any further divisions.



Moving on to the right parallelepiped of size $5 \times 9 \times 10$, we need to consider the planes parallel to the faces of the parallelepiped which create unit cubes, and the segments into which the diagonal is divided by these planes. However, much like the rectangle of size 5×10, we can ignore the side of length 5 (5 is a divisor of 10, and therefore the four planes perpendicular to the side of length 5 will not create any further divisions of the diagonal). Therefore, the diagonal will be divided by $8 + 9$ planes: we conclude that the diagonal will be divided into 18 parts, each of which is contained in a separate unit cube.