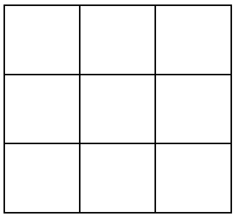
Mediterranean Youth Mathematical Championship (MYMC)
Rome, July 20, 2017

Early afternoon round – Last stage

RE2A.

In a 3×3 table, a positive integer is written in each cell so that the sum of each row and each column is odd. Of the nine numbers written in the cells, how many can be even?



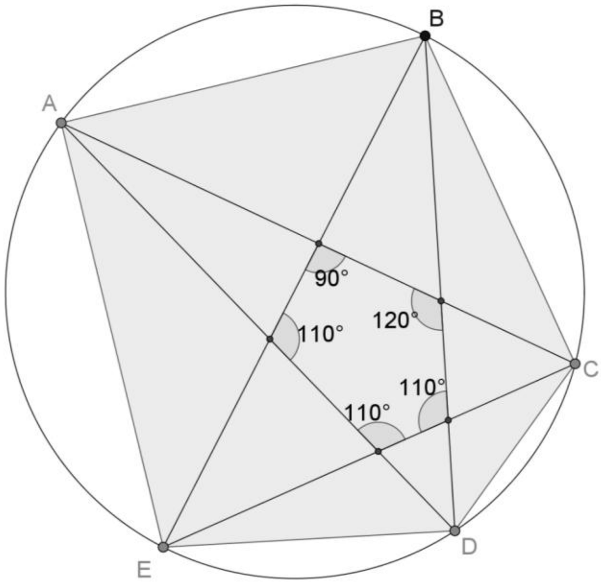
A) 0, 2, 4, 6, or 8
B) 0, 2, 4, or 6
C) 0, 4, 6, or 8
D) 0, 4, or 6
E) 0, or 4

Solution

The answer is D). First, we note that the sum of all the numbers written in the table will be odd and, therefore, we have an odd number of odd numbers, which means we have an even number of even numbers. It is also clear that a table without any even numbers meets our condition. Second, we also know that none of the rows can contain three even numbers, which means the table can never contain eight even numbers. Third, if there is an even number in the table, there must be another in the same row and another in the same column, which means that we can never have just two even numbers in the table. Finally, it is easy to construct examples with four or six even numbers (in the latter case, the odd numbers lie along a diagonal).

RE2B.

A pentagon $ABCDE$ is inscribed in a circle; its diagonals form another pentagon, as shown in the figure. The sizes of the angles of the internal pentagon are as indicated in the figure. Work out the size of the angles of the pentagon $ABCDE$.



Solution

The answer is: $\widehat{\boldsymbol{A}}$**=90° ;** $\widehat{\boldsymbol{B}}$**=100° ;** $\widehat{\boldsymbol{C}}$**=120° ;** $\widehat{\boldsymbol{D}}$**=130° ;** $\widehat{\boldsymbol{E}}$**=100° .** There are several ways to answer this question. Considering the five triangles with a vertex at the points A, B, ... and a side of the internal pentagon as a base, we can find the size of the angles DAC, EBD, ... With the theorem of angles subtended by the same arc, we see that the angle EAB is equal to the sum of the angles $EBD + DAC + CEB$. Hence, we find 90°, 100°, 120°, 130°, and 100°.

RE2C.

Alice chooses five distinct positive integers. She does not want to reveal the numbers. However, she agrees to give all of their pairwise sums: $17, 20, 28, 14, 42, 36, 28, 39, 25$, and 31. Find Alice's chosen numbers and list them in descending order.

Solution

The answer is **25, 17, 14, 11,** and **3**. Let $a > b > c > d > e$ be the five chosen numbers. In the list of their pairwise sums, each number appears four times—therefore,

$$a + b + c + d + e = \frac{1}{4}(17 + 20 + 28 + 14 + 42 + 36 + 28 + 39 + 25 + 31) = 70.$$

The largest sum must be $a + b = 42$ and the smallest must be $d + e = 14$, which means that $c = 70 - 42 - 14 = 14$. Furthermore, $a + c$ is the second largest sum, which implies that $a = 39 - 14 = 25$, and analogously $e = 17 - 14 = 3$. In this way, we can find $b = 42 - 25 = 17$ and $d = 14 - 3 = 11$.

GE2A.

Take eight normal dice and glue them together in such a way as to form a cube. If on each one of the six faces of the cube we add all numbers of the dice which are visible, we obtain the following six values:

15, 14, 19, 18, 11, and 19. Calculate the number obtained by adding the values of the hidden faces of the eight dice.

Solution

The answer is **72**. The sum of all the faces of the eight dice is $21 \cdot 8 = 168$. The sum of the values on the visible faces is $15 + 14 + 19 + 18 + 11 + 19 = 96$. Therefore, the sum of the hidden faces of the dice is $168 - 96 = 72$.

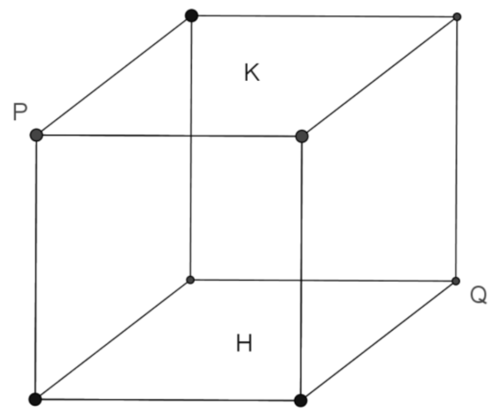
GE2B.

Let there be a sequence of positive integers $x_0, x_1, x_2, \ldots$ where $x_k = x_{k-1} + 2x_{k-2}$ for all $k \geq 2$. If $x_0 = 2$ and $2x_6 - x_5 = 130$, find x_1.

Solution

The answer is $x_1 = \mathbf{2}$. Working backwards, we get $2x_6 - x_5 = 2(x_5 + 2x_4) - x_5 = x_5 + 4x_4 = (x_4 + 2x_3) + 4x_4 = 5x_4 + 2x_3 = \ldots = 31x_1 + 34x_0$, from which $31x_1 + 34 \cdot 2 = 130$, $31x_1 = 62$, and therefore $x_1 = 2$.

GE2C.



From the cube shown in the figure we remove:

- the pyramid with the square H as its base and the point P as its vertex
- and the pyramid with the square K as its base and the point Q as its vertex.

The remaining part of the cube is:

A) a triangular pyramid;

B) a square-based pyramid;

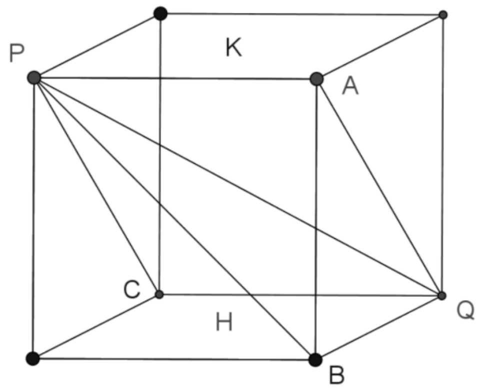
C) made up of two triangular pyramids;

D) made up of two square-based pyramids;

E) made up of a triangular pyramid and a square-based pyramid.

Solution

The answer is **C)**.



One way of visualising the situation is the following. Divide the cube into two parts with the plane $APCQ$. From the half of the cube containing the base H, remove the square-based pyramid with base H and vertex P, which leaves the triangular pyramid $PBQA$. The situation is analogous when we remove the second square-based pyramid from the other half of the cube.

We can also consider their volumes: we know that if the length of an edge is 1, each of the two pyramids with bases H and K described in the question have volume $\frac{1}{3}$, whereas each of the two pyramids that remain have volume $\frac{1}{6}$ (the area of the base AQB is $\frac{1}{2}$). Note that the diagonal PQ is an edge shared by all four described pyramids (the two described in the question and the two described in the solution).