Mediterranean Youth Mathematical Championship (MYMC)

Rome, July 20, 2017

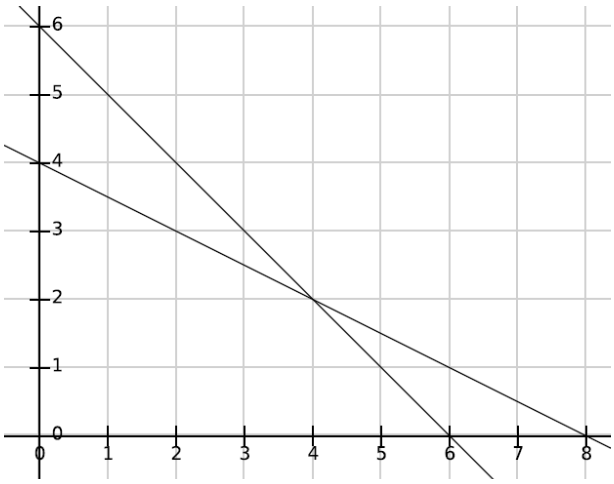
Early afternoon round – Intermediate stage

RE3A.

Find the pair of positive integers (x, y) such that $x + 2y \leq 8$ and $x + y \leq 6$, and $2y - x$ has the highest possible value.

Solution

The answer is $(\mathbf{1},\mathbf{3})$. We need to maximise the function $f(x, y) = 2y - x$over the pairs of positive integers found in the quadrilateral shown below, with vertices at the coordinates $(0, 0), (6, 0), (4, 2),$ and$(0, 4)$:



Exactly 11 pairs of positive integers are found within this quadrilateral: $(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (4, 1), (4, 2),$ and $(5, 1)$. To find the answer, all we need to do is find the pair with the highest y and lowest x values.

RE3B.

Let a and b be two positive integers, each less than 100, such that 121 is the area of the rectangle formed with the smallest of the integers as its base and their difference as its height. What is the value of $a + b$?

Solution

The answer is $\mathbf{33}$. If we assume $a \leq b$, then $121 = a(b - a)$ and we obtain the following table:

a	1	11	121
$b - a$	121	11	1
b	122	22	122

in which the second and fourth columns contradict the hypothesis that both a and b are less than 100. Therefore, the only possibility is that $a = 11, b = 22,$ and so $a + b = 33$. If we instead assume that $b \leq a$, we similarly find that $a = 22$ and $b = 11$, and again $a + b = 33$.

RE3C.

Let a convex quadrilateral $ABCD$ be placed on top of another convex quadrilateral $EFGH$, such that the vertices of $ABCD$ lie on the sides of $EFGH$. Every vertex of $ABCD$ lies on a different side of $EFGH$ and never coincides with a vertex of $EFGH$. Assume that if we fold the four non-covered triangles of quadrilateral $EFGH$ (along the sides *AB*, *BC*, *CD*, *DA*), we fill the quadrilateral $ABCD$ exactly (without overlap or leaving any holes). Then, necessarily:

A) $EFGH$ is a parallelogram

B) $ABCD$ is a rhombus and $EFGH$ is a rectangle

C) $EFGH$ is similar to $ABCD$

D) $EFGH$ has either perpendicular diagonals or a pair of parallel opposite sides

E) $ABCD$ has either perpendicular diagonals or a pair of equal consecutive sides

Solution

The answer is **D)**.

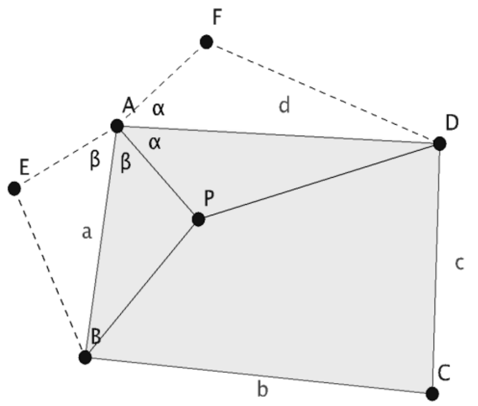


Figure 1

The triangle AEB and the folded-over triangle APB are symmetrical along the segment AB, from which we can deduce the equality of the two angles β shown in Figure1. We can show the same for the two angles α. Vertex A lies on the side EF of the external quadrilateral, meaning that E, A, and F are aligned. The sum of the angles in vertex A is therefore $2\alpha + 2\beta = 180°$, which means that $\alpha + \beta = 90°$. The same is true for all the angles of $ABCD$, which is therefore a rectangle. At this point, there are two possibilities: in the first case (Figure 2)

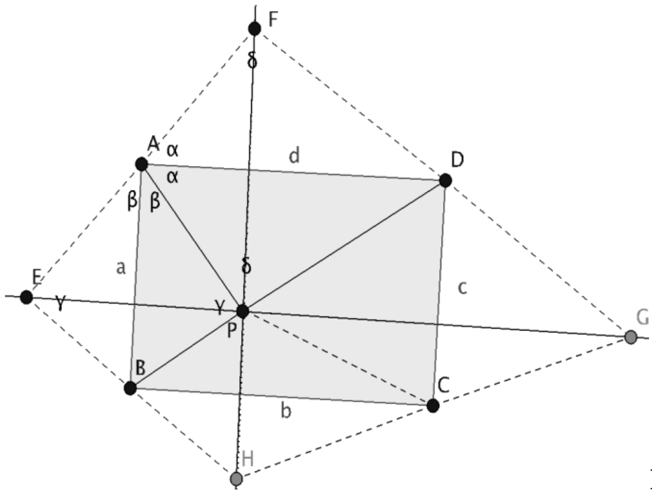


Figure 2

after folding over the triangles, the vertices of the quadrilateral meet at a point P. The four lines that connect P to E, F, G, and H coincide in pairs, such that they consist of two lines perpendicular to the sides of the rectangle $ABCD$ and are thus perpendicular to one another.
In the second case (Figure 3),

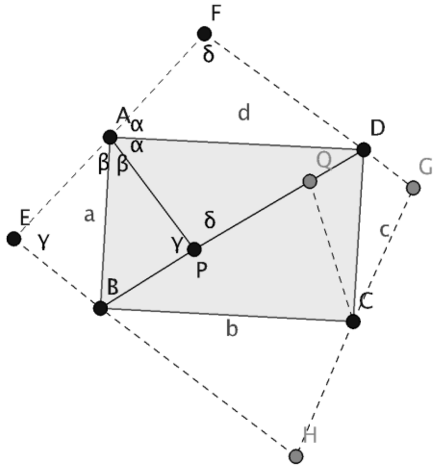


Figure 3

after having completed the fold, the vertices of the quadrilateral meet in pairs on a diagonal of the rectangle $ABCD$. With the notation used in the figure, and by what has previously been shown, we find that $\gamma + \delta = 180°$ (because P lies on the diagonal BD). However, γ and δ are also the angles in the vertices E and F, which are supplementary angles, and so the segments EH and FG are parallel.

GE3A.

If we calculate the product $(x-1)(x-2)(x-3)(x-4)(x-5)(x-6)(x-7)$, what will be the coefficient of x (that is, the linear coefficient)?

A) 13068

B) -5040

C) 28

D) -144

E) 40320

Solution

The answer is **A)**. The terms of the polynomial consist of all of the possible products obtained by choosing one term from each of the seven binomials. Now, to obtain the linear term, x, we need to pick x from one of the binomials and the numerical terms from all of the other binomials. If we pick x from the first binomial, the other terms will be $-2, -3, -4, -5, -6$, and -7, which means that the coefficient of this term will be $2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7$, which can be written as $\frac{7!}{1}$. If, instead, we pick x from the second binomial, multiplying the numerical terms gives us the coefficient $1 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7$, that is, $\frac{7!}{2}$. Proceeding in this way, the remaining x terms will have coefficients $\frac{7!}{3}, \frac{7!}{4}, \ldots, \frac{7!}{7}$. Adding together all of these coefficients, we get $7!\left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}\right) = 5040 \cdot \frac{363}{140} = 13068$.

GE3B.

Let r be a non-integer real number and $[r]$ its integer part ($[x]$ is the largest integer less than or equal to x). What is the number $[r]$, knowing that $r[r] = 2017$?

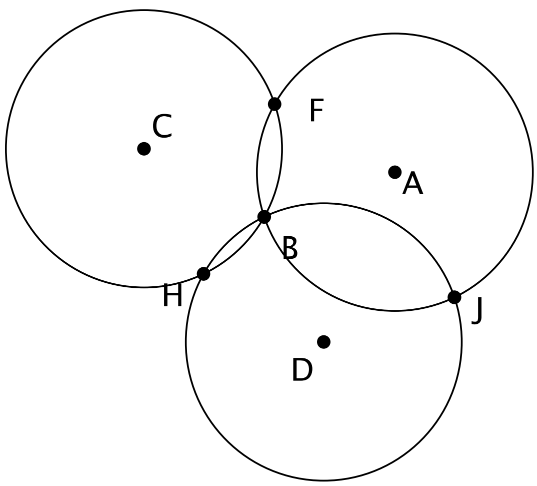
Solution

The answer is $\mathbf{-45}$. Because $44^2 = 1936 < 2017 < 2025 = 45^2$ and therefore $44 < \sqrt{2017} < 45$, to find the answer we simply need to check which of the following numbers the integer $[r] = \frac{2017}{r}$ can be:

$$-45, -44,\ 44,\ \text{or } 45.$$

GE3C.

Let there be three circles of the same radius of length 4, each passing through the same point B and none of them tangent to another. Let A, C, and D be the centres of these three circles, and let F, H, and J be the intersections (which do not coincide with B) of the circles with one another, as shown in the figure.



Assume, as in the figure, that the polygon P with vertices C, H, D, J, A, and F is convex. If the area of the triangle HJF is 18, what is the area of P?

Solution

The answer is **36**.

The area of P is the sum of the areas of the three rhombi $HDBC$, $DJAB$, and $BAFC$, and is therefore equal to double the area of the triangle formed by the circles' centres, ACD. In particular, HD is parallel and congruent to BC, and, by transitivity, to FA. The quadrilateral $HDAF$, which has two equal and parallel opposite sides, is a parallelogram. Therefore, we conclude that HF is parallel and congruent to AD. By the same reasoning, we can show that CD is parallel and congruent to FJ, and HJ is parallel and congruent to CA. The triangle ACD and the triangle HJF (which we know the area of) are congruent, because their sides are congruent to one another, and therefore they have the same area. It follows that the area of P is twice the area of the triangle HJF.

NB. The hypothesis that the length of the radius is 4 plays no part in this solution.