Mediterranean Youth Mathematical Championship (MYMC)

Rome, July 20, 2017

Early afternoon round – First stage

RE1A.

Let n be a positive integer with exactly five positive integer divisors (including 1 and the number itself). How many positive integer divisors does n^2 have?

A) 9

B) 12

C) 15

D) 25

E) The answer depends on the specific number n.

Solution

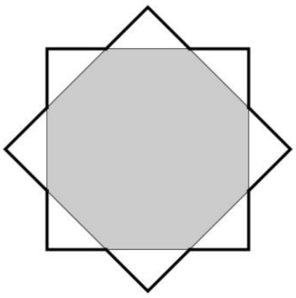
The answer is **A)**. In general, if $n = p^a \cdot q^b \cdot \ldots$ (where p, q are primes), the number of divisors of n is

$$(a+1) \cdot (b+1) \cdot \ldots$$

Therefore, if n has five divisors, n is necessarily of the type p^4, with p prime (for example, $n = 2^4 = 16$). Hence the number $n^2 = p^8$ has nine divisors.

RE1B.

Consider the eight-pointed star figure below, formed by the union of two squares with the same centre and with sides of the same length, such that the first square can be obtained by rotating the second square by an angle of 45° around its centre.



If the area of the shaded regular octagon is equal to 1, the area of the eight-pointed star is:

A) $\sqrt{2}$

B) 2

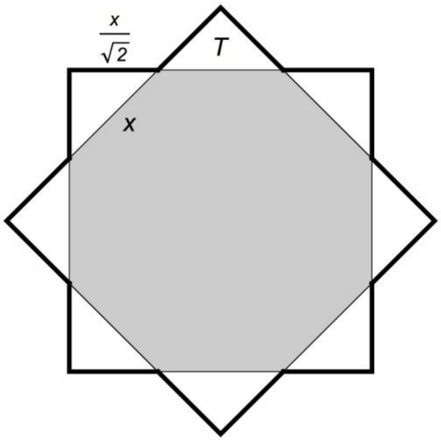
C) $1+\sqrt{2}$

D) $\sqrt{1+\sqrt{2}}$

E) $1+\frac{1}{\sqrt{2}}$

Solution

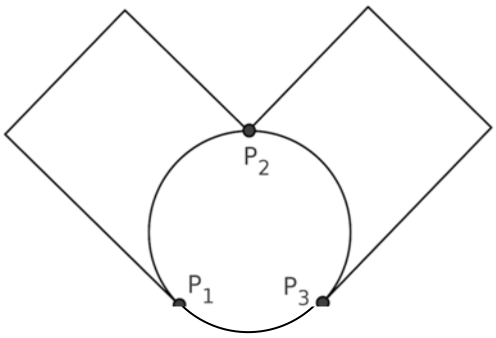
The answer is **A)**.



Let x be the side length of the shaded octagon and L the side length of the starting squares. Then let T be the area of each of the small triangles (that is, the points of the star) and A the area of the eight-pointed star. Using $T = \frac{x^2}{4}$ and $L = (1+\sqrt{2})x$, we can write the area of the octagon as $1 = L^2 - 4T = x^2\left(\left(1+\sqrt{2}\right)^2 - 1\right) = 2x^2(1+\sqrt{2})$, so that $x^2 = \frac{1}{2(1+\sqrt{2})}$. The area of the eight-pointed star is therefore $1 + 8T = 1 + 2x^2 = 1 + \frac{1}{1+\sqrt{2}} = \sqrt{2}$.

RE1C.

The king of Atlantis has a garden in which the paths form the following figure:



The king claims that he is able to walk from P_1 toP_3 in several different ways, going along each path exactly once. In how many ways can he do this?

Solution

The answer is **16**. Three paths meet at the point P_1, four paths meet at the point P_2, and again three paths meet at the point P_3. To go from P_1 to P_3, the king will therefore need to walk past P_1 one more time, past P_2 two times, and past P_3 only once (before reaching it again at the end of his route). When he first leaves from point P_1, the king has three possible paths, but when he passes P_1 again, he has only one possible path. Similarly, the first time the king reaches P_2, he can take three possible paths, but the second time he can only take one. Lastly, the first time he reaches P_3 he can take two possible paths, and the next time there are none left to take (as he has arrived at the end). In total, there can be at most $3 \cdot 1 \cdot 3 \cdot 1 \cdot 2 = 18$ possible ways. But, in fact, the number of ways is less than that, because choosing as the initial path either one of the two paths going from P_1 to P_2 there are 6 possible ways for each path of reaching P_3, while choosing as the initial path the one going from P_1 to P_3, the king will have just four ways of completing his walk.

Another point of view is the following. Walking around the whole garden once without stopping requires that the king go through the following points (initial and end points excluded): P_1 once, P_2 twice (not consecutively), and P_3 once. Hence he has four possible routes: $P_1P_2P_3P_2P_1P_3$, $P_1P_2P_3P_1P_2P_3$, $P_1P_2P_1P_3P_2P_3$, $P_1P_3P_2P_1P_2P_3$. Moreover, the first time the king walks from P_1 to P_2 (or from P_2 to P_1) he can choose between two possibilities, and also the first time he walks from P_3 to P_2 (or from P_2 to P_3) he can choose between two possibilities. Hence $4 \cdot 2 \cdot 2 = 16$ choices altogether.

GE1A.

A palindromic number is a number that remains the same when its digits are reversed, namely which reads the same backwards as it does forwards. Which of the following numbers is the largest palindromic prime smaller than 1000?

A) 929

B) 919

C) 989

D) 959

E) 949

Solution

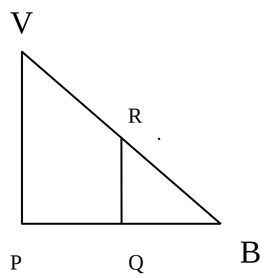
The answer is **A)**. Note that $989 = 23 \cdot 43$, $959 = 7 \cdot 137$, and $949 = 13 \cdot 73$ are not prime numbers. Both 929 and 919 are primes.

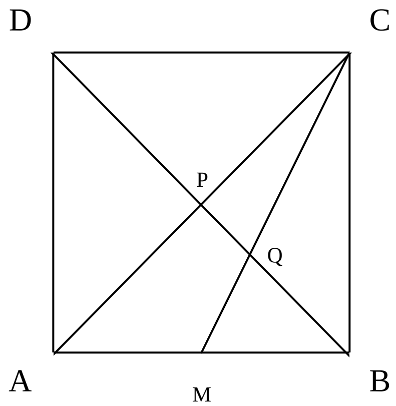
GE1B.

The pyramid of the pharaoh MYMC is 60 metres tall, with a square base $ABCD$ of sides 102 metres. The vertex of the pyramid lies vertically above the centre of its square base. The Giza meridian cuts the base along the segment CM, where M is the midpoint of the side AB. Find the maximum height (in metres) of the points of the pyramid which lie vertically above the segment CM.

Solution

The answer is **40**. The maximum height is the length of the segment QR shown in the figure on the left,





in which V is the vertex of the pyramid, whereas P and Q are as shown in the figure on the right.

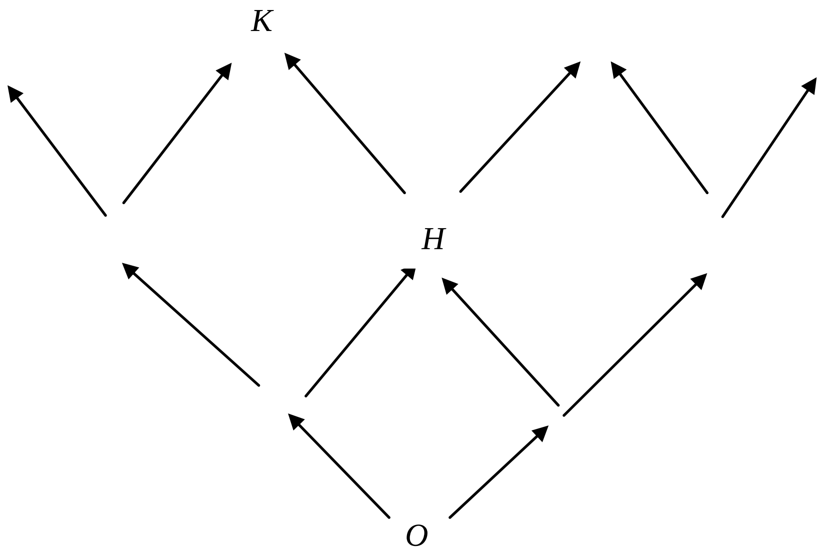
Letting a be the length, in metres, of the sides of the base, the similarity between the triangles BMQ and CDQ implies that $\overline{DQ}:\overline{BQ} = a : \frac{1}{2}a$, which means that $\left(a\sqrt{2} - \overline{BQ}\right):\overline{BQ} = 2$, from which $a\sqrt{2} - \overline{BQ} = 2\overline{BQ}$ and $\overline{BQ} = \frac{1}{3}a\sqrt{2}$.

It follows that $\overline{BQ} = \frac{2}{3}\left(\frac{a\sqrt{2}}{2}\right) = \frac{2}{3}\overline{BP}$. Because the two triangles BPV and BQR are also similar, we conclude that $\overline{QR} = \frac{2}{3}\overline{VP} = \frac{2}{3}\,60 = 40$.

NB. The hypothesis that $a = 102$ plays no part in this solution.

GE1C.

Consider the following picture:



Mary starts at point O at the bottom and moves upwards by following the arrows. Whenever she finds herself at a point from which two arrows originate, the probability that she takes the right path is p and the probability that she takes the left path is $1 - p$, where $0 < p < 1$ (the value of p is the same at all points). Suppose that the probability x of reaching point H is the same as the probability of reaching point K. Which of the following statements is true?

A) $0 < x < \frac{1}{4}$

B) $\frac{1}{4} < x < \frac{1}{2}$

C) $\frac{1}{2} < x < \frac{3}{4}$

D) $\frac{3}{4} < x < 1$

E) The probability of reaching point H is always different to the probability of reaching point K.

Solution

The answer is **B)**. The probability of reaching point H is $2p(1-p)$. The probability of reaching point K is $3(1-p)^2p$. Assuming these are the same and simplifying (remembering that $0 < p < 1$), we find that $p = \frac{1}{3}$ and therefore $x = \frac{4}{9}$.