# Mastermind a combinatorial approach

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**Laura Lamberti (Liceo Augusto Righi, Roma)**
**Francesca Tovena (Università Tor Vergata, Roma)**

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# The didactic relevance of games in math

The game is a **didactic tool** that helps to involve and excite children.

In every game there are rules that must be understood and applied and there is a goal to be achieved.

Students, captivated by the game, are naturally interested in reflecting on rules and procedures and are willing to put their skills into motion.

This means that gamifying math eases the transmission of mathematical concepts .

In particular the Mastermind game allows the introduction of **elements of set theory**, **logic** and **combinatorics** and favors their experimentation.

The search for winning strategies helps the teacher to talk about **decision-making algorithms**.

# Educational pourposes of the activity

The  activity aimed to improve

**1. Partecipation**: gamifying math is a very powerful mean to capture the interest of the students.
It is very natural to explore mathematical structures relying under the game rules and procedures.
To solve the game kids are naturally moved to discuss their attempts.

**2. Language**: because of the CLIL methodology.

**3. Logical-mathematical**: In the specific the activity was focused on Mastermind.
This game allows training the mind by improving its logical skills
improves the ability to formulate hypothesis and deduce consequences
It provides a valuable tool for developing combinatorics and set theory skills

**4.Argument**: especially in the game between teams, it develops the ability to support one's own hypotheses
in comparison with others, to seek solution strategies; it favors debates among the students.

**5. Socialization:** the activity was done during the pandemic: it made it possible
to transform a very difficult period for children a bit lighter.

# Mastermind: the game

Essentially Mastermind is a two players game: a **code-maker** who chooses a code that is kept secret, while the other player, the decoder (**code-breaker**) must guess it by making a certain number of attempts.

The **secret code** is made up of an ordered sequence of colours. In the commercial version of the game, the code consists of 4 coloured pegs, each of which selected from a set of 6 different elements. Repetitions are allowed.

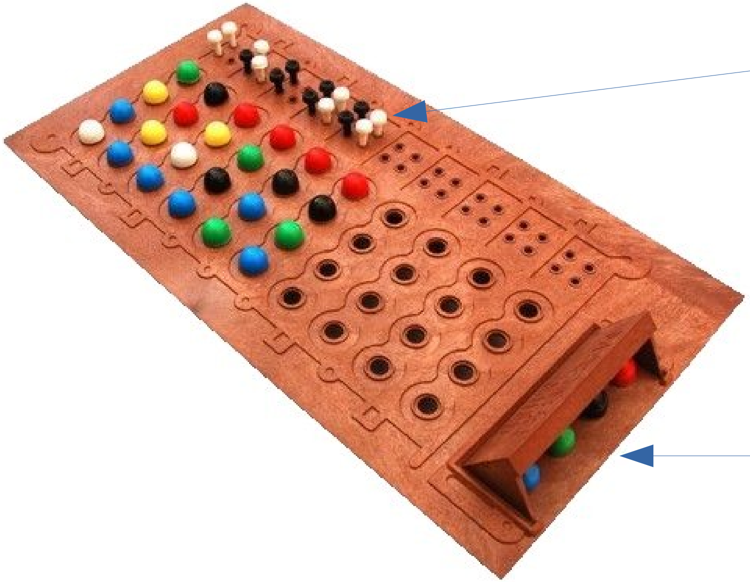
You win if you guess the secret code in a **number of moves lower than the one established as the maximum**.

# Mastermind: the game

The code-breaker tries guessing the code by placing **coloured pegs** in a row of a perforated table.

The code-maker gives a feedback providing clues as to the correctness of any attempt.

Alongside each attempt, the player who selected the secret code writes down an answer using **black and white tiny dots**. This feedback of the code-maker is built following the subsequent rules:

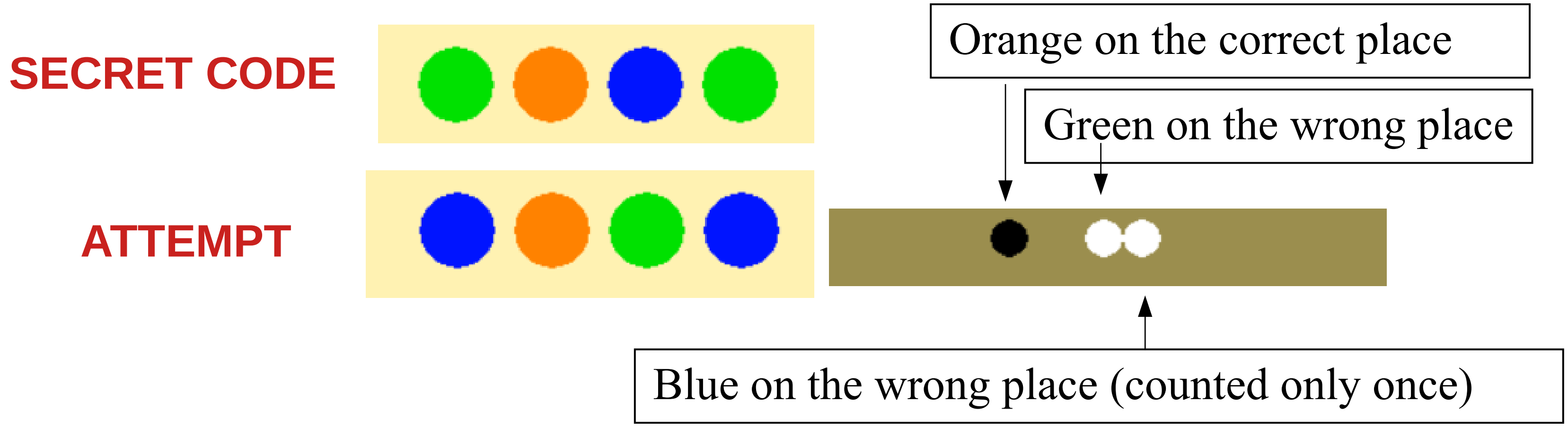


1. for each correct colour in the correct place in the attempt, a **black dot** is inserted in the answer.
2. instead the **white dots** are used to communicate the presence of an element of the right colour but which is not in the correct place
3. any peg is considered only once in the analysis

SECRET CODE

# Mastermind: the game

Example:



# Mastermind: the game

Several attemps and feedbacks leading to the code breaking

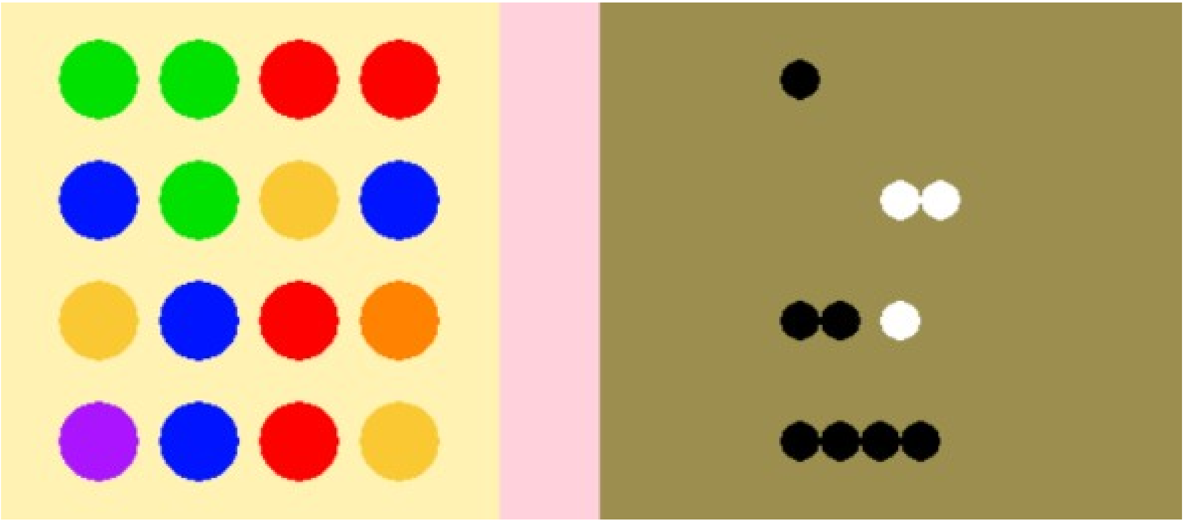
**ATTEMPTS** **FEEDBACKS**

1

2

3

SECRET CODE 4

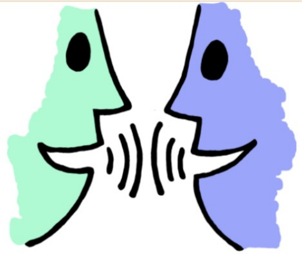


# Description of the didactic activity 1. CLIL

The educational project was carried out between March and May 2021 with a second class of scientific high school (**fifteen years old students**) in Rome (Italy).

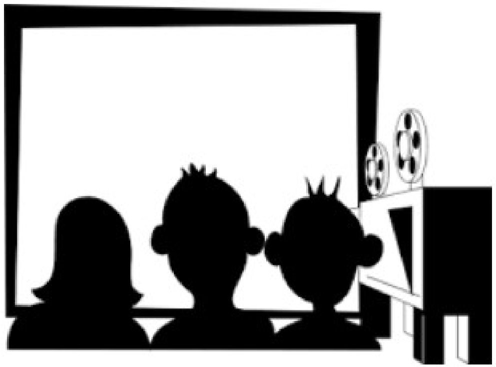
It was carried out partly during distance-learning times and partly face-to-face since the school-time was reduced because of the pandemic. The hours spent at school were **at least twelve**.

The **CLIL methodology** was used to introduce the subject:
the foreign language used was English;
the project was entirely realized by the math teacher who is not a native English speaker.
All the students are Italian native speakers, but have already achieved a quite good level in English.



# Description of the didactic activity 1. CLIL

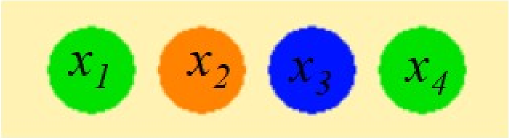
In order to get the **Listening** language skill while working on the game, some **videos** explaining the rules of the game and showing some procedures were played and discussed.



Students learned the instructions: they began to **play the game also using game implementations freely available online** and experimenting possible winning strategies.

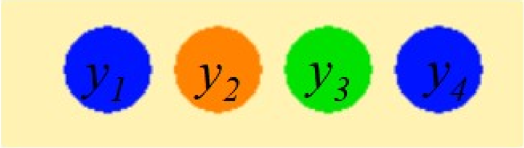
The English terms used to specify the rules of the game were shared during this segment

# Description of the didactic activity 2. math




The second part of the activity was dedicated to the **mathematical formalization** of the problem.

The **secret code** can be represented by the sequence $x_1,x_2,x_3, x_4$
while the guess trial is $y_1,y_2,y_3,y_4$




The elements $x_i$, $y_k$ are chosen in a set of 6 elements (the colours).

The number of the possible codes (dispositions of 6 elements of class 4) is **$6^4$=1296**. (assuming the standard version)

The code-breaker wins when he puts the right coloured pegs in the right places.
That means when his attempt satisfies the condition:

$$x_i=y_i \quad \text{for} \quad i=1...4.$$

The **feedbacks** can be seen as ordered couples $(n_{blacks}, n_{whites})$ formed by the comparison between the guess and the secret code.

In order to understand the winning strategy a **simplified case** was proposed and studied.

# Description of the didactic activity 2. 3x3 case

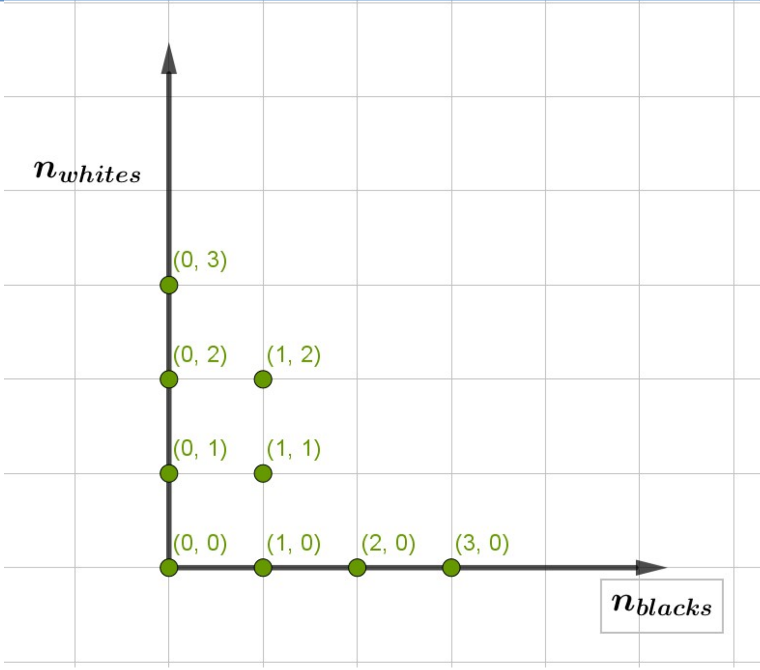
Suppose that **the secret code is made up of 3 elements** chosen among a set of **3 colours {A,B,C}**.
Repetitions are however allowed.

The possible dispositions are **$3^3$=27**.

The first attempt is compared to the secret code and is related to the received feedback.

In this simplified version there are 9 possible feedbacks to a played attempt :

$$n_{whites} + n_{blacks} \leq 3$$



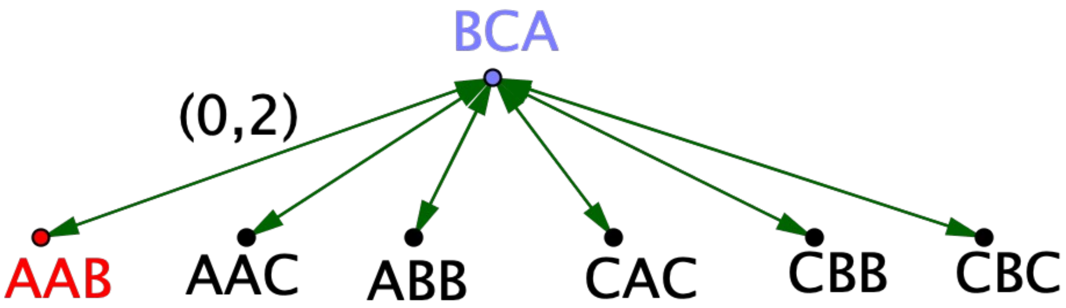

The total set of dispositions can be **partitioned** by the use of the answers **comparing each disposition to the attempt .**

# Description of the didactic activity 2. 3x3 case

The feedback ($n_{blacks}$, $n_{whites}$) represents the "**compatibility relation**" between the code and the player's attempt.

Example: Suppose that the secret code is **AAB**
and the attempt is **BCA**
The feedback will be **(0,2)**

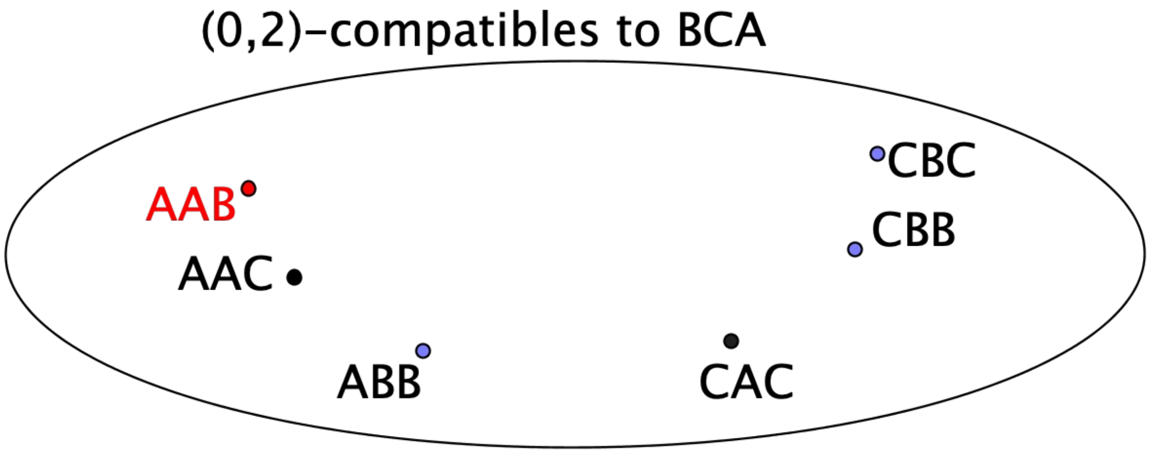
**For simmetry reasons all the dispositions which have the same answer as compared to the attempt BCA can be the secret code (and the secret code is one of them)**




**This implication was quite difficult to be transmitted to the class!**

# Description of the didactic activity 2. 3x3 case

Among the 27 dispositions, the subset {AAB, AAC, ABB, CAC, CBB, CBC} of the dispositions "(0,2)-**compatibles**" with **BCA** **has to contain the secret code**.

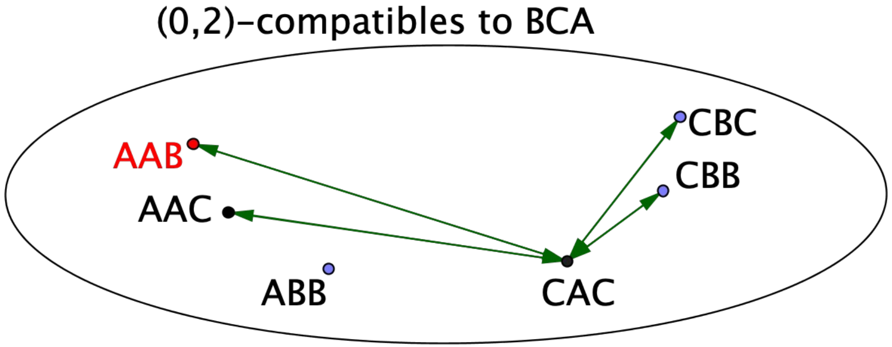


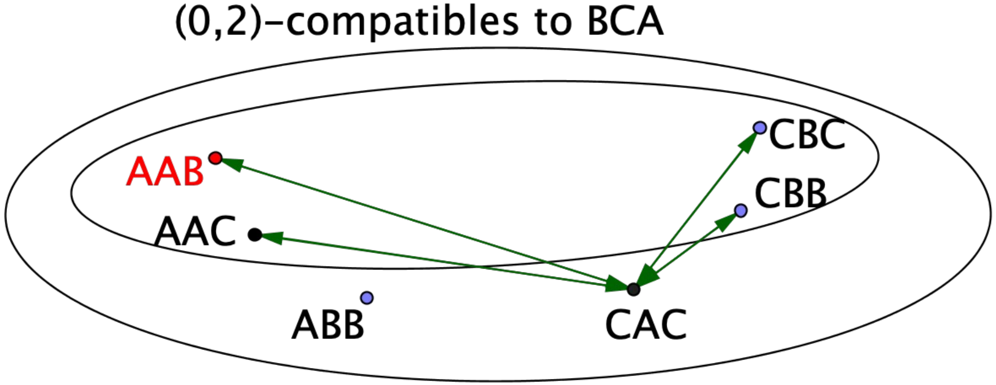

As a strategy, it is possible to select a **new attempt** among the elements of the subset of compatible codes just identified.

# Description of the didactic activity 3. 3x3 case

For example, suppose we have selected CAC as the **second attempt.**
The feedback with respect to the secret code AAB is (1,0).

In the subset of the dispositions compatibles with BCA, the new answer allows to select the elements that are also (1,0)-compatible with CAC



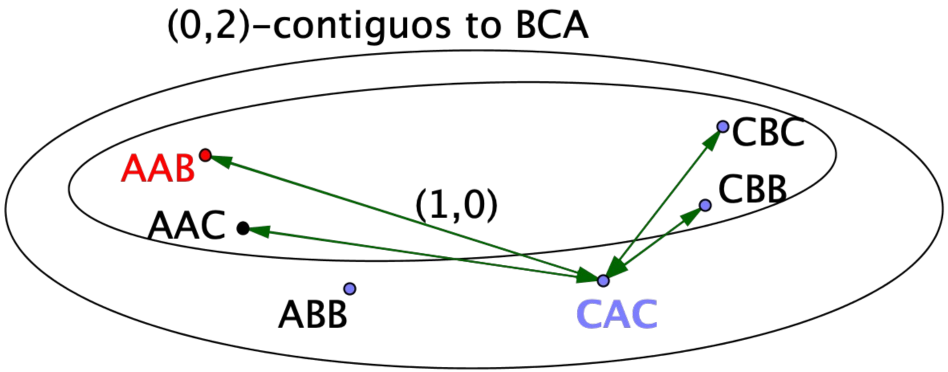

So, we have a smaller subset, containing the secret code

# Description of the didactic activity 3. 3x3 case

So, it is possible to search for the secret code by **repeating the** following **procedure:**

- you choose an attempt in the subset of compatible codes with all the answers already obtained (at the beginning, you use all the arrangements)

- use the answer to determine a proper subset formed by all the codes 'compatible' with that answer




**At each step**, the number of elements in the subset of ‘compatibles’ codes **is reduced**: then **the procedure ends in a finite number of steps**, determining the secret code.

# Description of the didactic activity 2. 3x3 case

**First attempt BCA**

**Feedbacks Subsets /elements**

| | | | |
|---|---|---|---|
| **(3,0)** | **BCA** | | |
| **(2,0)** | **BCB** | **BCC** | **BBA** |
| | **ACA** | **BAA** | **CCA** |
| **(1,0)** | **AAA** | **BBB** | **CCC** |
| **(1,1)** | **BAB** | **BBC** | **CAA** |
| | **CCB** | **ACC** | **ABA** |
| **(1,2)** | **BAC** | **ACB** | **CBA** |
| **(0,2)** | **CBB** | **CBC** | **CAC** |
| | **ABB** | **AAB** | **AAC** |
| **(0,3)** | **ABC** | **CAB** | |
| **(0,1)** | empty | | |
| **(2,1)** | empty | | |
| **(0,0)** | empty | | |

With respect to the answer it is possible to **partitionate the set of the total dispositions** into several subsets each of which is composed by the dispositions which have in common the same answer with respect to the attempt that generated the partition.

The maximum number of elements of these subsets is 6. That means that in the worst case we will reduce the cardinality from 27 to 6.

# Description of the didactic activity 2. 3x3 case

Mastermind is a **combinatorial game**.

The mathematical approach to the game is based on **working on sets**, **disjoint subsets**, **partitions**, **intersections of subsets**, **dispositions** and **iterations**.

It is important to notice that the partition of the initial set is not unique
since every **different attempt** chosen from among the provisions can produce a **different partition**.
The cardinalities of the subsets are also different.

For each possible attempt and for each possible feedback, students determined the number of compatible codes.
Then they calculated the maximum of the values obtained for each attempt:
they interpreted this maximum as the number of codes remaining to be analysed **'in the worst possible case'**.
At each step, the attempt that gets the **minimum of the maximum** value is chosen.

It is a special case of a recurrent procedure in game theory, called **min-max algorithm**.

# Description of the didactic activity 2. 3x3 case

The **min-max algorithm**:

Since the aim of the code-breaker player is to win the game in the minimum number of attemps,
his winning strategy will be directed to the construction of subsets whose cardinalities are the lowest.

Most of sum-zero games like Mastermind can be approached by using a similar min-max algorithm.

For each partition thus obtained in the set of possible dispositions it is necessary
to **evaluate the cardinality K of the largest subset (worst case)**.
Since the partition depends on the starting attempt the algorithm chooses the partition
which produces subsets with the **least value of K**.
In this way the number of possible arrangements is the minimum at each step,
while the solution is achieved always in a finite number of steps.

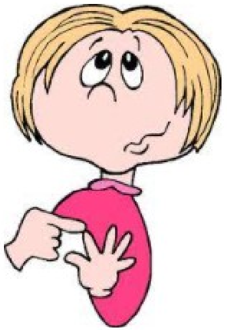
To apply the min-max algorithm is necessary to
**analyse the behavior of the starting guess** with respect to the maximum cardinality of the produced subsets.
At every move the starting point must be compared to all the other elements in order to classify each element.

This step can take a quite huge time to be solved.

# Description of the didactic activity 2. 3x3 case

The **min-max algorithm**:

which is the optimum choise of the attempts?



An investigation on the **partitions produced by the guesses** was carried out by the students.
It is just a **counting and comparing problem.**

The attempts were divided into three categories:
the one formed by all different colours (ABC),
those containing a couple (AAC for instance)
and those formed by the same colour (AAA for instance).

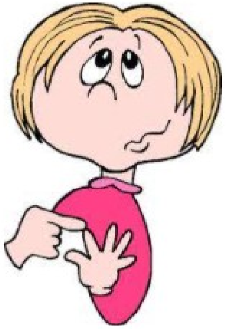
(Because of the simmetry, the behaviour was the same for each element belonging to the same group).

# Description of the didactic activity 2. 4x6 case

The **min-max algorithm in the standard version 4x6**:

To set the more suitable first attempt in the standard version (6-4)
an analogous investigation was performed counting the elements of each subset formed by the partitions.

In this case the dispositions were divided into five categories.
For each of these classes it has been obtained the **maximum cardinality** of the produced subsets .



The analysis of the results shows that

**starting with an attempt formed by two couples**
**such as AABB is the better choise,**

as Knuth suggested in his paper.

# Conclusions

The proposed activity was found to be effective in various aspects:

**Partecipation**: gamifying math is a very powerful mean to capture the interest of the students.
It is very natural to explore mathematical structures relying under the game rules and procedures.
To solve the game kids are naturally moved to discuss their attempts, apply rules, making hypothesis and assumptions.
Lessons are very enjoiable, not boring at all. Students often continue to think about the game after the schooltime, sometimes, once home, they propose the game to their parents and so are forced to explain rules and strategies and definitely to talk about math.

**Language**: even if the CLIL activity was related to a single part of the whole didactic activity,
it became evident that the subject teacher is also able to exploit opportunities for developing language skills.
Watching a video, introducing the specific terms to express the rules, working with attractive images
like those of the codified videogame are very good instruments to go beyond the difficulties
met starting speaking in a foreing language.

# Conclusions

The proposed activity was found to be effective in various aspects:

**Logical-mathematical**: In the specific the activity was focused on Mastermind.
This game allows training the mind by improving its logical skills, lets the student get used to critical reasoning, improves the ability to formulate hypothesis and deduce consequences, forces the student to analize and discharge hypotheses if they are in disagreement with the results achieved;
the game allows to show how sometimes apparently negative results can be of great help for the solver, as it happens quite frequently in math.
It provides a valuable tool for developing combinatorics and set theory skills

**Argument**: especially in the game between teams, it develops the ability to support one's own hypotheses in comparison with others, to seek solution strategies; it favors debates among the students.

**Socialization:** thinking about the period when the activity was done, during the pandemic, it made it possible to make a very difficult period for children a bit lighter.

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## Grazie!

**Laura Lamberti (Liceo Augusto Righi, Roma) lauralamberti@liceoaugustorighiroma.it**
**Francesca Tovena (Università Tor Vergata, Roma) ftovena@gmail.com**

www.crf.uniroma2.it

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